Killing fields and conservation laws

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June 13, 2011

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- Illustrate a connection between

Killing Fields		**Conservation Laws**
of primitive maps		as elements of the characteristic cohomology
$\mathrm{d}B_\lambda + [\psi_\lambda, B_\lambda] = 0$		$\varphi \in \bar{H}^1(M^{(\infty)})$

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- Use the connection above to completely determine the conservation laws as elements of the characteristic cohomology for

$$u_{z\bar{z}} = -f(u) \quad \text{where} \quad u : \mathbb{C} \to \mathbb{R}.$$

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- Raise questions about what information is stored in conservation laws, thought of as cohomology classes

Nonlinear Poisson equation

In order to encode

$$\frac{\partial^2 u}{\partial z \partial \overline{z}} = -f(u),$$

as an EDS with independence condition, let $M = J^1(\mathbb{C}, \mathbb{R}) = \mathbb{C} \times \mathbb{R} \times \mathbb{C}$ have coordinates (z, u, u_0) and define the differential forms

$$\zeta = \mathrm{d}z$$

$$\eta_0 = \mathrm{d}u - u_0 \zeta - \overline{u}_0 \overline{\zeta}$$

$$\psi = -\frac{\sqrt{-1}}{2}(\zeta \wedge \mathrm{d}u_0 - \overline{\zeta} \wedge \mathrm{d}\overline{u}_0 + 2f \,\mathrm{d}z \wedge \mathrm{d}\overline{z}).$$

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Independence condition: $\zeta \wedge \overline{\zeta} \neq 0$.

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If $A : M^{(\infty)} \to \mathbb{C}$ then $\mathrm{d}A \equiv e_{-1}(A)\zeta + \overline{e_{-1}}(A)\overline{\zeta}$ modulo $\mathrm{I}^{(\infty)}$.

Nonlinear Poisson equation: $\mathbb{S}^1$-symmetry and weighted-degree

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is a symmetry of

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for $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

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$$F_\lambda^* \mathrm{I}^{(\infty)} = \mathrm{I}^{(\infty)}$$

Conservation laws as elements of the characteristic cohomology

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Group of Conservation Laws:

$$\overline{H}^1 := H^1\left(\Omega^*(M^{(\infty)})/\mathcal{I}^{(\infty)}, \overline{\mathrm{d}}\right)$$
$$= \frac{\left\{\varphi \in \Omega^1(M^{(\infty)}) \mid \mathrm{d}\varphi \in \mathcal{I}^{(\infty)}\right\}}{\mathrm{I}^{(\infty)} + \mathrm{d}\Omega^0(M^{(\infty)})}$$

Analogy with Chern classes

$$\begin{array}{ccc} \mathbb{C}^k \longrightarrow V & \rightsquigarrow & c_i(V) \in H^{2i}(X,\mathbb{Z}) \\ \quad\;\; \downarrow \pi & & \\ \quad\;\; X & & \end{array}$$

To each complex vector bundle we associate Chern classes that measure the *twisting of the bundle.*

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$c_i(V) = f^*(c_i(U_k))$ for $f : X \to Gr(k,\infty)$ a classifying map.

Characteristic cohomology induces de Rham cohomolgy on solutions

Solution *EDS*

$$(N, \iota) \xrightarrow{\quad \iota \quad} (M, \mathcal{I})$$

$$[\iota^*(\varphi)]_{dR} \in H^1_{dR}(N) \longleftarrow [\varphi]_{CC} \in \overline{H}^1$$

The space of generating functions

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From now on, assume that $f_u \neq \alpha f + \beta$ for any $\alpha, \beta \in \mathbb{R}$.

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Joint work with Oliver Goertsches (Selecta 2011)

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7. *If* $f_{uu} = \beta f$ *for* $\beta \in \mathbb{R}$ *(e.g.* $f = \sin$, $f = \sinh$*) then* $\dim_{\mathbb{C}}(V_{2n+1}) = 1$ *for all* $n \in \mathbb{Z}$.

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- In what sense have **all** the conservation laws or symmetries been found?
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- Is there a general formulation that will work for $u : \mathbb{C} \to \mathbb{R}^m$—the Toda-field equations—when $m > 1$?

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Killing fields lead to a recursion that satisfies the first two and will very likely accommodate the third criterion

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$(G^{\mathbb{C}}, \tau, \sigma)$ is a *k*-**symmetric space**

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where $\mathfrak{g}_j$ is the eigenspace of τ with eigenvalue μ^j, for some primitive k^{th}-root of unity μ.

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Let N be a simply connected Riemann surface. The map $\phi : N \to \mathrm{G}/\mathrm{K}$ has a framing $F : N \to \mathrm{G}$.

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Primitive maps and the Toda-field

Bolton, Pedit, Woodward (1995) show that, when $K = T^m$, there exist coordinates and a frame for which

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There are two stages of the recursion that require a relative Poincare lemma. This relative Poincare lemma follows from the vanishing result $V_{2n} = 0$.

Killing fields

based loop algebra associated to $\mathfrak{g}^{\mathbb{C}}$

$$\mathcal{L}(\mathfrak{g}^{\mathbb{C}}) =$$

$$\left\{ B_\lambda \in \mathfrak{g}^{\mathbb{C}}[[\lambda, \lambda^{-1}]] : \text{ if } B_\lambda = \sum_n B^n \lambda^n \text{ then } B^0 = 0 \right\}$$

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Definition

A **formal Killing field** for the family of flat connections ψ_λ on the Riemann surface N is a map $B_\lambda : N \to \mathcal{L}^{\sigma,\tau}(\mathfrak{g}^{\mathbb{C}})$ satisfying

$$\mathrm{d}B_\lambda + [\psi_\lambda, B_\lambda] = 0.$$

Killing fields and infinitesimal symmetries

Burstall, Ferus, Pedit, Pinkall (1993) introduce Killing fields of Harmonic maps into symmetric spaces as a way to package infinitesimal symmetries (Jacobi fields)

Killing fields unpacked

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$$\partial B^j + [\psi_0', B^j] + [\psi_{-1}, B^{j+1}] = 0$$
$$\bar{\partial} B^j + [\psi_0'', B^j] + [\psi_1, B^{j-1}] = 0.$$

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We find that $\mathcal{E}_{(\mathrm{G}^{\mathbb{C}},\sigma,\tau)}(B^{kn}) = 0$ for $B^{kn} \in \mathfrak{g}_0$.

Killing fields and conservation laws

For any pair of functions $P, Q \in \Omega^0(N, \mathfrak{g}_0)$ define

$$\varphi_{P,Q} = -\sqrt{-1}J(\kappa(P, \mathrm{d}Q) - \kappa(Q, \mathrm{d}P)) \in \Omega^1(N, \mathbb{C})$$

where κ is the Killing form of $\mathfrak{g}^{\mathbb{C}}$.

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The $\mathfrak{g}_0$-components of Killing fields give rise to conservation laws for primitive maps (when $\mathsf{K} = T^m$).

Terng and Wang (2004) gave a similar formula for conservation laws of the U/K-systems using Killing field-like objects.

The recursion for $f_{uu} = \alpha f_u + \beta f$

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b^n is a polynomial in $u_0, \ldots, u_{2n+1}$.

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Recursively construct

$$
\begin{aligned}
f^n &= \sqrt{-1}e^{\frac{u}{2}}(b^n_{-1} - u_0 b^n) \\
r^n &= \frac{1}{3\sqrt{2}}e^{-\frac{u}{2}}(f^n_{-1} + \frac{1}{2}u_0 f^n) \\
s^n &= -\frac{1}{\sqrt{2}}e^{-\frac{u}{2}}r^n_{-1}.
\end{aligned}
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Again, define a one-form

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Summary: **There is a level six recursion derived from the Killing field equation; requires $V_{2n} = 0$.**

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Using the seeds

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for the order six recursion, implies that $V_{2n+1} \cong \mathbb{C}$ for $n \in \mathbb{Z}$ and $n \neq 1, -2$.

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This agrees with the definition of finite type given by Pinkall and Sterling.

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A linear embedding—the spectral curve?

Suppose that $\iota : \mathbb{C} \to M^{(\infty)}$ is a doubly periodic solution for the lattice $\Lambda \subset \mathbb{C}$, of finite type n.

Let $\{[\varphi_j]\}$ be a basis for $\overline{H}^1$ such that the generating function P_j of φ_j satisfies $P_j = u_{j-1} + \cdots$ for $j \geq 1$.

Let $\bar{H}^1_n := \{[\varphi_j] \in \bar{H}^1 \mid j \leq n\} \cong \mathbb{R}^{2n}$

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The angles determining this linear embedding should contain geometric information.

A linear embedding—the spectral curve?

Conjecture

$\hat{\iota}^* : H_1(\mathbb{C}/\Lambda, \mathbb{R}) \to (\bar{H}^1_n)^*$ descends to a map

$$\hat{\iota}^* : H_1(\mathbb{C}/\Lambda, \mathbb{R})/\Lambda^* \to \mathbb{R}^{2n}/\Gamma \subset Jac(X_u)$$

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Idea: Conservation laws define an *extended* Abel-Jacobi map

Global approach

Global conservation laws for minimal surfaces $N \to \mathbb{S}^3$ might lead to a maximally linear embedding

$$N \to Jac(N) \hookrightarrow T^{2n}$$

even if N is a compact Riemann surface with genus> 1, thus extending the spectral curve type construction to higher genus domains.

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Pedit has suggested the same possibility based on the multiplier spectral curve.