Conservation Laws and Generalized Isometric Embeddings

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Outline

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II. Generalized Isometric Embedding Problem

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III. Two Fundamental Motivations

IV. Results

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V. Solving Strategy

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Conservation Laws

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1.

$$\mathcal{F}$$

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Genralized Isometric Embedding Problem (F. Hélein)

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$\mathcal{M}^m$

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$$\begin{array}{c} \Big\downarrow \\ \mathcal{M}^m \end{array}$$

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$$\begin{tikzcd} \mathbb{V}^n \arrow[d] \\ \mathcal{M}^m \end{tikzcd}$$

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$$
\begin{tikzcd}
g, \quad \mathbb{V}^n \arrow[d] \\
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\end{tikzcd}
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\begin{tikzcd}
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$$\begin{array}{ccc} g,\nabla \quad \mathbb{V}^n & \overset{\Psi\,?}{\hookrightarrow} & \mathcal{M}^m \times \mathbb{R}^{N^n_{m,p}} \\ \mathrm{d}_\nabla \phi = 0 \ \big\downarrow & & \big\downarrow \\ \mathcal{M}^m & & \mathcal{M}^m \end{array}$$

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- The ingredients of the generalized isometric embedding are :

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- The problem is trivial when $n = 1$.

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$$\begin{tikzcd}
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$\lambda, \mu, \nu = 1$ to m

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- $\phi = E_i \phi^i$.

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What happens locally ?

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What happens locally ?

$i, j, k = 1$ to n $\qquad$ $A, B, C = 1$ to N

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$\mathcal{M}^m$ $\qquad\qquad$ $\mathcal{N}^n$

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$$\mathcal{M}^m \xrightarrow{\quad u \quad} \mathcal{N}^n$$

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$$\begin{array}{ccc} & & \mathrm{T}\mathcal{N}^n \\ & & \downarrow \\ \mathcal{M}^m & \xrightarrow{u} & \mathcal{N}^n \end{array}$$

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A harmonic map u produces the ingredients $(u^*\mathrm{T}\mathcal{N}^n, \mathcal{M}^m, g, \nabla, \star\mathrm{d}u)_{(m-1)}$ of the generalized isometric embedding problem.

The Keystone !

The Keystone ! Noether's Theorem !

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$$\mathcal{M}^m \xrightarrow[\text{harmonic}]{u} \mathcal{S}^n$$

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$$\Psi(M, v) = (M, u \times v)$$

The Generalized Isometric Embedding Problem's Goal

To show the existence of the analogeous of conservation laws when there are no symmetries for a system of partial differential equations.

Results

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Construction of local conservation laws by generalized isometric embedding of vector bundles, arXiv :0804.2608 (2010), to appear in **Asian J. of Math**.

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Theorem ($(\mathbb{V}^{\mathbf{n}}, \mathcal{M}^{\mathbf{m}}, \mathbf{g}, \nabla, \phi)_{\mathbf{m-1}}$ case)

Let $\mathbb{V}^n$ be a real analytic n-dimensional vector bundle over a real analytic m-dimensional manifold $\mathcal{M}^m$ endowed with a metric g and a connection ∇ compatible with g. Given a non-vanishing covariantly closed $\mathbb{V}^n$-valued differential $(m-1)$-form ϕ, there exists a local generalized isometric embedding of $\mathbb{V}^n$ in $\mathcal{M}^m \times \mathbb{R}^{n+\kappa^n_{m,m-1}}$ over $\mathcal{M}^m$, where $\kappa^n_{m,m-1} \geqslant (m-1)(n-1)$ such that the image of ϕ is a conservation law.

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Let $\mathbb{V}^2$ be a real analytic 2-dimensional vector bundle over a real analytic m-dimensional manifold $\mathcal{M}^m$ endowed with a metric g and a connection ∇ compatible with g. Given a non-vanishing covariantly closed non-degenerate $\mathbb{V}^2$-valued differential 1-form ϕ, there exists a local generalized isometric embedding of $\mathbb{V}^2$ in $\mathcal{M}^m \times \mathbb{R}^{n+\kappa^2_{m,1}}$ over $\mathcal{M}^m$, where $\kappa^n_{m,m-1} \geqslant 1$ such that the image of ϕ is a conservation law.

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Let $\mathcal{M}^4$ be an oriented real analytic 4-dimensional manifold endowed with a metric (actually a conformal structure is enough). Consider a real analytic vector bundle $\mathbb{V}^3$ of rank 3 over $\mathcal{M}^4$, endowed with a Riemannian metric g, g-compatible connection ∇, and a anti-self-dual covariantly closed $\mathbb{V}^3$-valued differential 2-form ϕ. There exists then a local generalized isometric embedding Ψ of $\mathbb{V}^3$ into $\mathcal{M}^4 \times \mathbb{R}^{3+\kappa^3_{4,2,ASD}}$, where $\kappa^3_{4,2,ASD} \geqslant 4$, such that $\Psi(\phi)$ is a local conservation law.

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Frédéric Hélein, *Manifolds obtained by soldering together points, lines, etc.* in Geometry, topology, quantum field theory and cosmology, C. Barbachoux, J. Kouneiher, F. Hélein, eds, collection Travaux en Cours (Physique-Mathématiques), Hermann 2009, p. 23–43. **arXiv :0904.4616.**

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The generalized isometric embedding EDS is

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$$\mathcal{K}^n_{m,p} := \{(\mathcal{R}^i_{j;\lambda\mu}) \in \wedge^2(\mathbb{R}^n) \otimes \wedge^2(\mathbb{R}^m) |\, \Omega^i_j \wedge \phi^j = 0, \forall i = 1 \text{ to } n\}.$$

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$$(\mathcal{G}^n_{m,p})^i_{j;\lambda\mu} : H_{i\lambda}.H_{j\mu} - H_{i\mu}.H_{j\lambda} = \mathcal{R}^i_{j;\lambda\mu}.$$

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6. Studying the problem for a structural group $G \subset O(n)$.

Thank you very much for your attention

Merci infiniment de votre attention