# Small(er) is Beautiful: Twistor Space Constructions in Submanifold Geometry

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June 2011

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The EDS $\mathcal{I}$ on $F$ for minimal surfaces is generated by $\omega^3$ and

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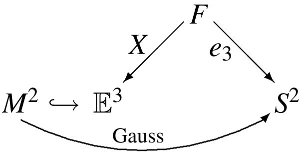


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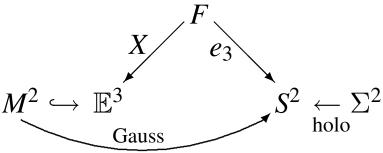


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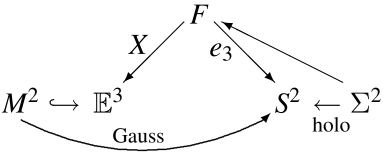


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Conversely: let $\Sigma^2$ be an abstract Riemann surface with local coordinate $z$, and $w = g(z)$ a map to $\mathbb{C} \subset S^2$. Then an integral of $\mathcal{I}$ is constructed by choosing a second holomorphic function $f(z)$ such that $\eta = f(z)\,\mathrm{d}z$, then integrating $\mathrm{d}X = (e_1 - \mathrm{i}e_2)\eta$, giving

$$X(z) = \mathrm{Re} \int (1 - g^2, \mathrm{i}(1 + g^2), 2g) f \mathrm{d}z.$$

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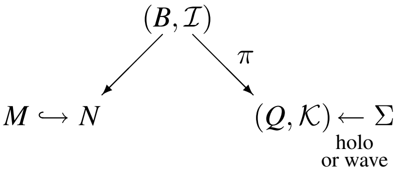


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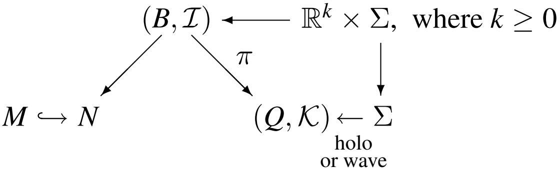


We seek a system $\mathcal{K}$ defined on a quotient manifold $Q$, such that
–integrals of $\mathcal{K}$ are easily obtained (e.g., via C-R or wave eqns.)
–integrals of $\mathcal{K}$ can be lifted to integrals of $\mathcal{I}$
(optimal scenario: $\mathcal{I}$ is an integrable extension of $\mathcal{K}$)

Idea: Look for smaller systems inside $\mathcal{I}$ whose generators are semibasic for $\pi$, and which *drops* to the 'twistor space' $Q$.

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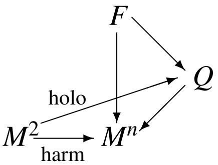
Twistor spaces for harmonic mappings (Eels-Salamon):

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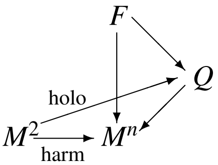


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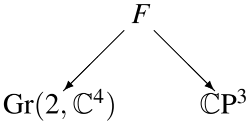
Twistor spaces for instantons on Minkowski space (Penrose):




Here, $\mathrm{Gr}(2, \mathbb{C}^4)$ is compactified complexified Minkowski space and $F$ is a flag manifold.

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–carries over to complex Pfaffian systems $I \subset T^*M \otimes \mathbb{C}$
–generalizes to any bundle $I \subset \Lambda^k V$ of $k$-forms semibasic for $\pi$

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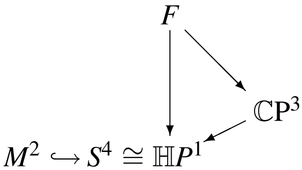


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Adapting frames along minimal $M$ such that $e_1, e_2$ are tangent, we get

$$\psi^a = H^a \eta, \quad \text{for some } H^a \in \mathbb{C}, \quad a = 3, 4$$

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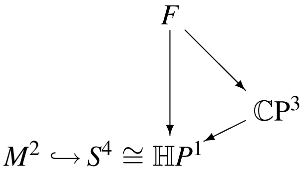


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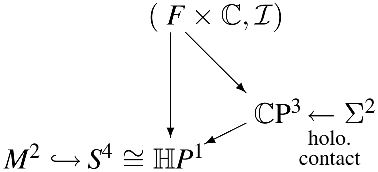


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Problem: Classify circular $M^4$'s that are ruled by $J$-closed 2-planes.

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Thm (Ionel & I) For each $p \in M$ there is a 6-dimensional subspace $F_p \subset \mathbb{R}^n$, such that $T_pM \subset F_p$, and an orthogonal complex structure $\mathbb{J}_p$ on $F_p$ (extending $J$) such that, for all $X \in T_pM$

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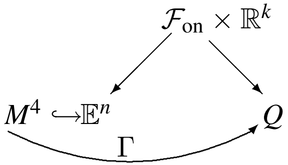
Let $Q = SO(n)/SO(2) \times U(2) \times SO(n-6)$ be the space of triples $(E, F, \mathbb{J})$. Then we can locally reconstruct $M$ from the image of the rank 2 holomorphic map $\Gamma : p \mapsto (E_p, F_p, \mathbb{J}_p)$.



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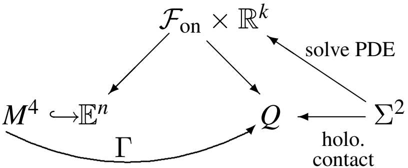

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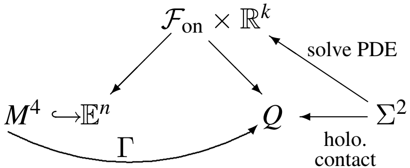


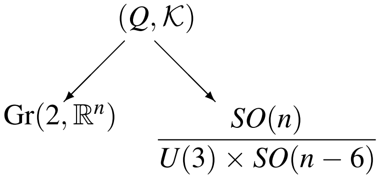

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Define a product "twistor space" $Q = Q_1 \times Q_2$, and determine how to lift pairs of curves to integrals of $\mathcal{I}$.

# Example: Flat Surfaces in $S^3$

The EDS $\mathcal{I}$ on $F \cong SO(4)$ for flat surfaces is generated by $\omega^3$ and

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Characteristic systems contain Frobenius systems

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Given a product embedding of $\Sigma = \mathbb{R} \times \mathbb{R}$ into $Q$, the pullback of $\mathcal{I}$ to $\pi^{-1}(\Sigma)$ is Frobenius.

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Each characteristic system contains a Frobenius system of rank $2n-1$, and a contact form which drops to the quotient $S^{2n-1}$. Given a product embedding of contact submanifolds into $Q$, the inverse image is an integral of $\mathcal{I}$.

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Questions
▷ Can we determine the topology of complete ruled austere 4-folds in $\mathbb{R}^6, \mathbb{R}^7, \ldots$?
▷ For a given symmetric space $G/H$, can we classify homogeneous EDS's $\mathcal{I}$ on $G$ that admit "useful" twistor spaces?