On Ribaucour transformations for surfaces

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Centre de Recherches Mathematiques, June 2011

Workshop on Moving Frames in Geometry

Ribaucour Transfomations (Classical definition)

M, $\tilde{M}$ **surfaces in** $\bar{M}^3(k)$ **without umbilic points,**

$\psi : M \to \tilde{M}$ **diffeomorfism such that:**

a) $\exp_p h(p)N(p) = \exp_{\psi(p)} h(p)\tilde{N}(\psi(p)), \ \forall p \in M$**;**

b) $M_0 = \{\exp_p h(p)N(p) | p \in M\}$ **is a surface in** $\overline{M}$**;**

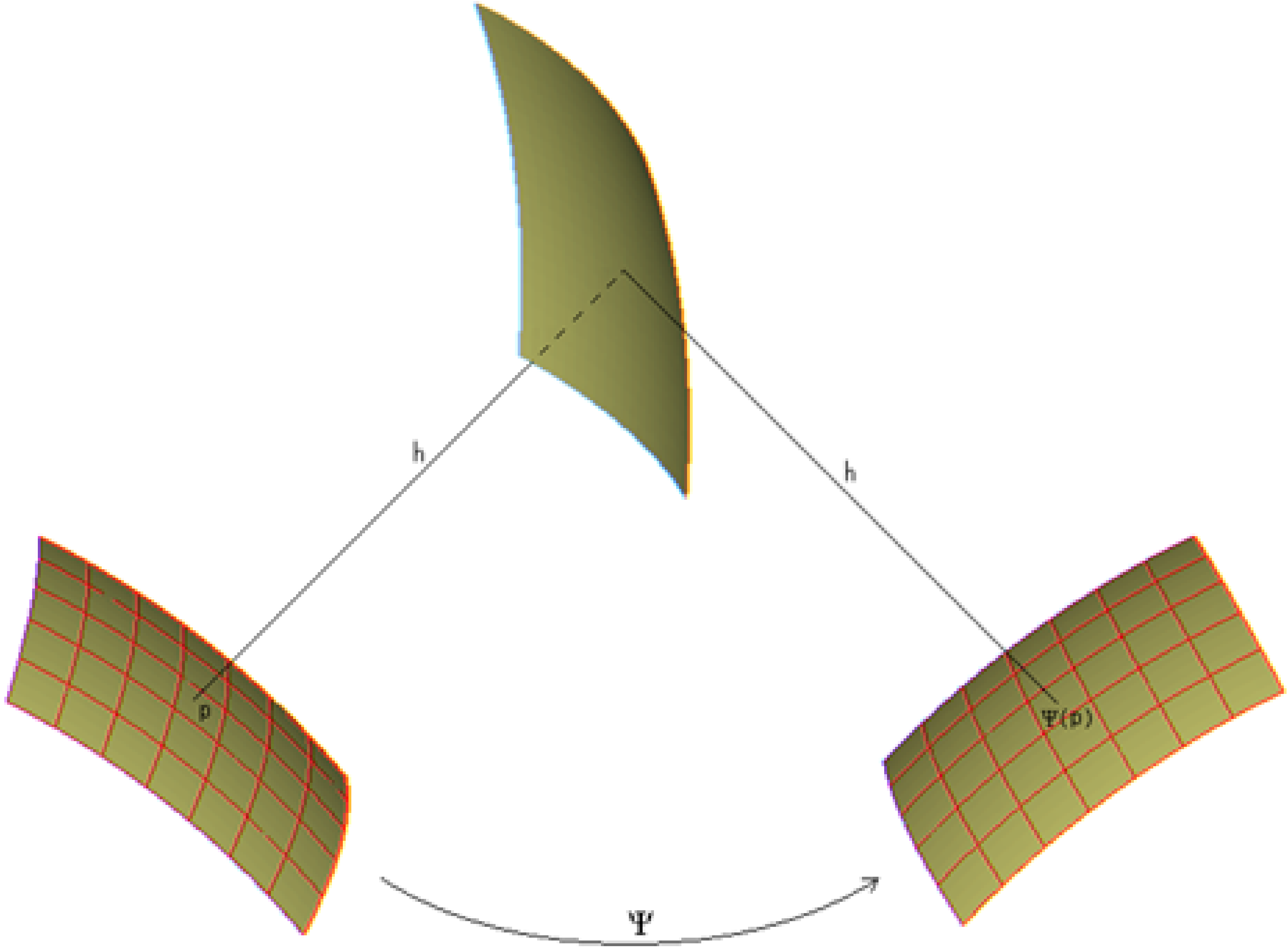
c) ψ **preserves lines of curvature.**

a) can be rewritten as

$$p + h(p)N(p) = \psi(p) + h(p)\tilde{N}(\psi(p)), \quad p \in M, \text{ if } \ k = 0,$$

and

$$h(p) = \begin{cases} \tan(\phi(p)), & \phi : M \to \left(0, \frac{\pi}{2}\right), \quad \text{if } \ k = 1, \\ \tanh(\phi(p)), & \phi : M \to \mathbb{R}, \quad \text{if } \ k = -1. \end{cases}$$



Ribaucour transformation in $\mathbb{R}^3$

Remarks

- **We only need the existence of an orthonormal frame of principal directions e_1, e_2 on M.**
- **Require $d\psi(e_1)$ and $d\psi(e_2)$ to be orthogonal principal directions on $\tilde{M}$.**
- **Higher dimensional generalization of RT, Corro ____ (2004).**
- **The hypersurfaces $\tilde{M}$ may differ according to the chosen frame (when the principal curvatures of M have multiplicity > 1).**

- **Ribaucour transformations (RT) between surfaces of constant Gaussian curvature, cmc or minimal surfaces were known since 1918 (Ribaucour, Bianchi).**
- **First examples of minimal surfaces using RT were obtained by Corro, Ferreira, ____, 2003.**
- **RT were extended to linear Weingarten (LW) surfaces in $\mathbb{R}^3$ by Corro, Ferreira, ____, 2003) and in space forms by ____, Wang (2006).**
- **These results provided an extension and a unified version of the classical results. We obtained several applications and geometric properties of RT.**
- **This talk is a survey of some results (2003 - 2011).**

Remarks:

- **$\tilde{\text{M}}$ may be locally associated to M by a Ribaucour transformation.**
- **We consider the hyperbolic three space as the submanifold of L^4**

$$I\!H^3 = \{x \in I\!L^4 | < x, x >= -1\}$$

with two connected components.

- **A Ribaucour transformation is equivalent to solving a non-linear PDE for h. This equation can be reduced to a linear system of diff. equations by considering $h = \Omega/W$.**

A characterization of Ribaucour transformations in $\bar{M}^3(k)$

Theorem A. **Let $M \subset \bar{M}^3(k)$ be a surface which admits an o.n. frame e_1, e_2 of principal directions. A surface $\widetilde{M}$ is locally assoc. to M, by a Ribaucour transf. $\Longleftrightarrow h = \frac{\Omega}{W}$ where Ω and $W \neq 0$ satisfy**

$$d\Omega = \sum_{i=1}^{2} \Omega_i \omega_i,$$

$$dW = \sum_{i=1}^{2} \Omega_i \omega_{i3},$$

$$d\Omega_i(e_j) = \Omega_j \omega_{ij}(e_j), \; i \neq j.$$

If X is a local parametrization of M then $\tilde{M}$ is parametrized by

$$\tilde{X} = \left(1 - \frac{2k\Omega^2}{S}\right) X - \frac{2\Omega}{S} (\nabla\Omega - W e_3) \quad \textbf{where } S = \sum_{j=1}^{2} (\Omega_j)^2 + W^2 + k\Omega^2$$

Linear Weingarten (LW) surfaces in $\bar{M}^3(k)$

$$\alpha + \beta H + \gamma(K - k) = 0, \qquad \alpha, \beta, \gamma \in R$$

H and K are the mean and Gaussian curvatures.

We say it is

hyperbolic when $\Delta := \beta^2 - 4\alpha\gamma < 0$

eliptic when $\Delta > 0$

$\Delta = 0$ characterizes the tubular surfaces.

In particular a surface is:

hyperbolic when $K - k = -1$

eliptic when $K - k = 1$, cmc or minimal.

Ribaucour transformations for LW surfaces

(Corro, Ferreira, _____, Wang)

Theorem B. Let M and $\widetilde{M}$ be regular surfaces in $\bar{M}^3(k)$ associated by a Ribaucour transformation. If Ω_i, Ω and W satisfy the **additional condition**

$$S = 2c(\alpha\Omega^2 + \beta\Omega W + \gamma W^2)$$

$c \neq 0$, α, β, γ are real numbers and $S = \Omega_1^2 + \Omega_2^2 + W^2 + k\Omega^2$. Then

$\widetilde{M}$ satisfies $\alpha + \beta\tilde{H} + \gamma(\tilde{K} - k) = 0$,

$$\Updownarrow$$

M satisfies $\alpha + \beta H + \gamma(K - k) = 0$.

Special Cases

- **cmc H surfaces**

$$\alpha = -H \neq 0, \quad \beta = 1, \quad \gamma = 0$$

the algebraic condition reduces to

$$S = 2c\Omega(-\mathbf{H}\Omega + \mathbf{W}),$$

and c **must satisfy** $\mathbf{c}(\mathbf{c} - \mathbf{2H}) - \mathbf{k} > \mathbf{0}$**.**

- **Minimal surfaces**

$$\alpha = \mathbf{0}, \quad \beta = \mathbf{1}, \quad \gamma = \mathbf{0}$$

the algebraic condition reduces to

$$\mathbf{S} = \mathbf{2c\Omega W}.$$

Let $M \subset \bar{M}^3(k)$ be LW surface.

Let e_1 and e_2 be an o.n. frame of principal directions.

If M satisfies $\alpha + \beta H + \gamma(K-k)$ then the RT is the integrable system:

$$d\Omega = \sum_{i=1}^{2} \Omega_i \omega_i,$$

$$dW = \sum_{i=1}^{2} \Omega_i \omega_{i3},$$

$$d\Omega_i = \Omega_j \omega_{ij} + \{(2c\alpha - k)\Omega - \beta c W\}\omega_i + \{c\beta\Omega + (2c\gamma - 1)W\}\omega_{i3}, \ i \neq j.$$

with initial condition satisfying

$$\Omega_1^2 + \Omega_2^2 + W^2 + k\Omega^2 = 2c(\alpha\Omega^2 + \beta\Omega W + \gamma W^2),$$

- **Generically we get a three parameter family of surfaces.**

Embedded planar ends in $\mathbb{R}^3$.

Theorem. **(Corro, Ferreira, ______) Consider** $\widetilde{X} : D \setminus \{p_0\} \to R^3$, $X : D \to R^3$ **minimal surfaces, locally assoc. by a RT such that** Ω **and** W **are defined on** D. **If** $S(p_0) = 0$, $\Omega(p_0) \neq 0$ **and** $S(p) \neq 0$, $\forall p \in D \setminus \{p_0\}$,

(a) **for any divergent curve** $\gamma : [0,1) \to D \setminus \{p_0\}$ **such that** $\lim_{t \to 1} \gamma(t) = p_0$ **the length of** $\widetilde{X}(\gamma)$ **is infinite.**

(b) $\widetilde{X}$ **has an embedded planar end at** p_0, **and** $\lim_{p \to p_0} \widetilde{N}(p) = N(p_0)$.

Proposition. **(Corro, Ferreira, ______) Consider the catenoid parametrized by**

$$\mathbf{X}(\mathbf{u_1},\mathbf{u_2}) = (\cos\mathbf{u_2}\cosh\mathbf{u_1},\ \sin\mathbf{u_2}\cosh\mathbf{u_1},\ \mathbf{u_1})$$

Up to rigid motions of R^3, a parametrized surface $\tilde{\mathbf{X}}_\mathbf{c}$ is a minimal surface, locally associated to X by a Ribaucour transformation as in Theorem B $\Longleftrightarrow$

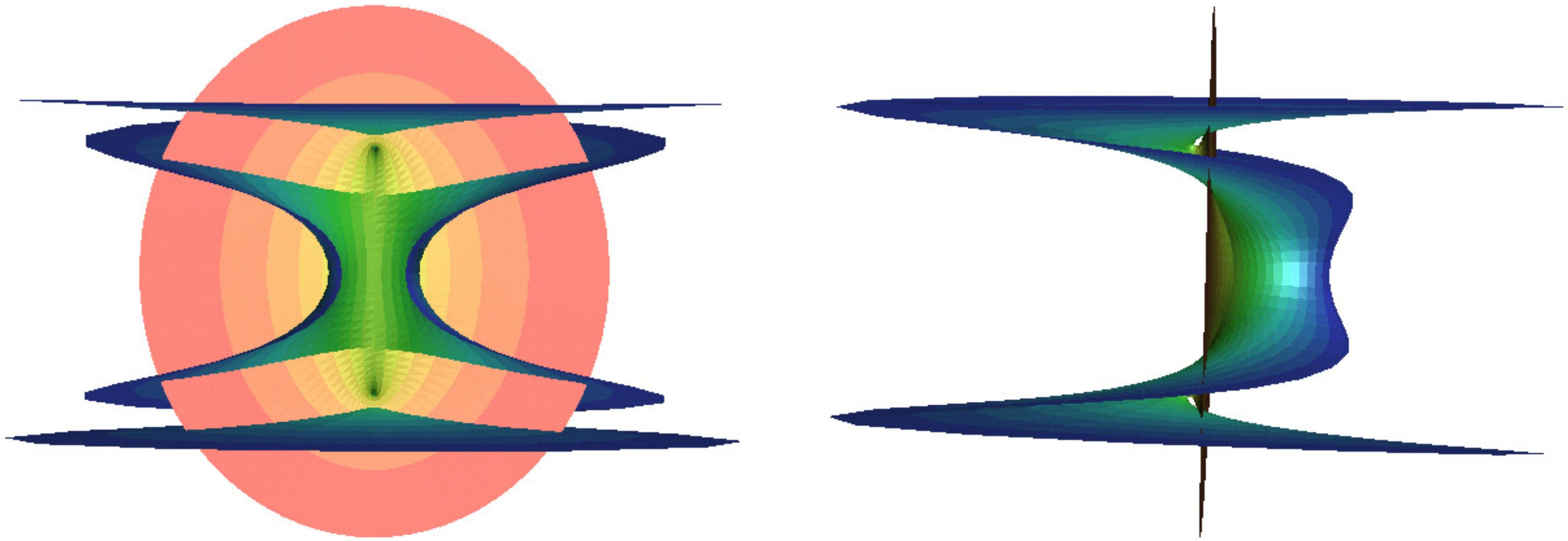
$$\tilde{\mathbf{X}}_\mathbf{c} = \mathbf{X} - \frac{\cosh\mathbf{u_1}}{\mathbf{c}}(\cos\mathbf{u_2},\sin\mathbf{u_2},\mathbf{0}) + \frac{\mathbf{1}}{\mathbf{c}(\mathbf{f}+\mathbf{g})}(\mathbf{f}'\mathbf{X}_{\mathbf{u_1}} - \mathbf{g}'\mathbf{X}_{\mathbf{u_2}})$$

where $\mathbf{c} \neq \mathbf{0}$, $\mathbf{f}(\mathbf{u_1})$ and $\mathbf{g}(\mathbf{u_2})$ satisfy

$$\mathbf{f}'' + (\mathbf{2c}-\mathbf{1})\mathbf{f} = \mathbf{g}'' - (\mathbf{2c}-\mathbf{1})\mathbf{g} = \mathbf{0}$$

and the initial conditions satisfy

$$(\mathbf{f}')^2 + (\mathbf{g}')^2 + (\mathbf{2c}-\mathbf{1})(\mathbf{f}^2 - \mathbf{g}^2) = \mathbf{0}.$$



$\mathbf{c = 1/2}$, *a family of complete minimal surfaces of genus zero and two ends. Each surface has one embedded planar end. The other end wraps around the catenoid infinitely many times.*

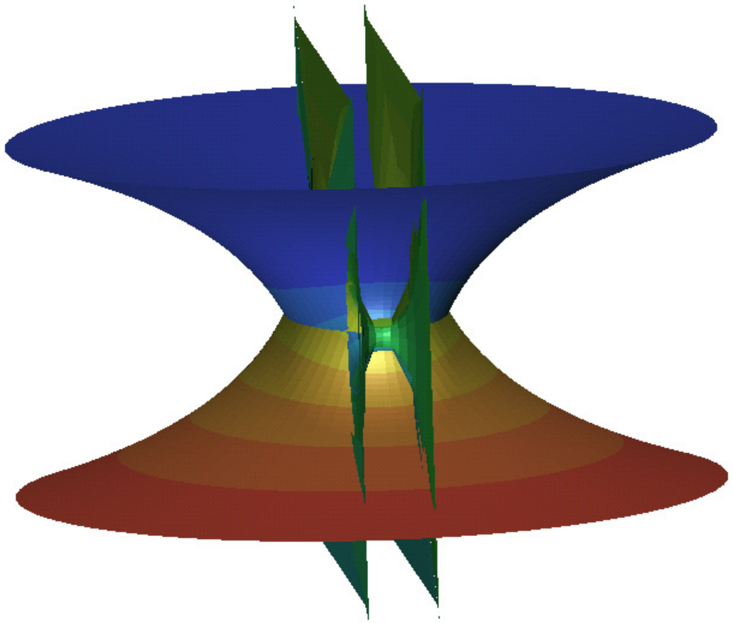
For $\mathbf{c \neq 0}$, $\mathbf{c < 1/2}$ and $\sqrt{\mathbf{1-2c}} = \mathbf{n/m}$ is an irreducible rational numbers, $n \neq m$.

- **We obtain a family of complete minimal surfaces modelled on a sphere punctured at $\mathbf{n+2}$ points, which depends on a parameter A.**
- **It has $\mathbf{n}$ embedded planar ends and 2 nonplanar ends of geometric index $\mathbf{m}$.**
- **The total curvature is $-4\pi(n+m)$.**
- **The parameter A affects the direction of the planar ends.**

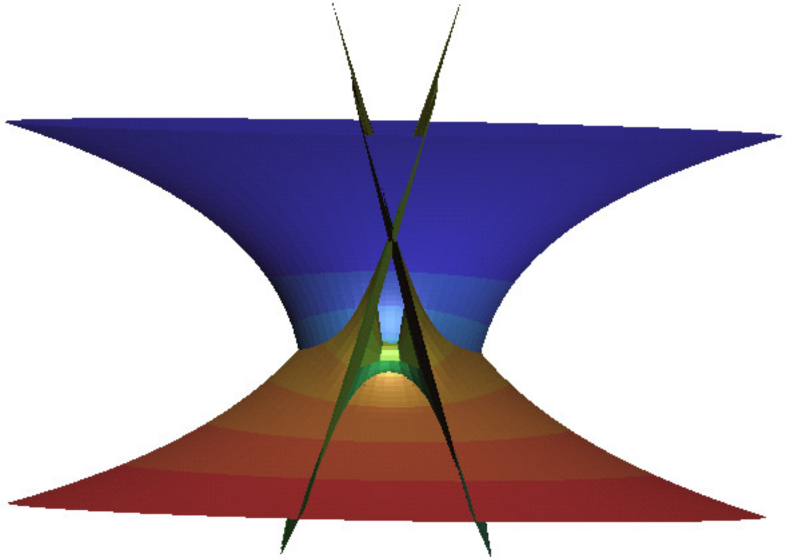
When $\mathbf{c > 1/2}$ or $\mathbf{0 \neq c < 1/2}$ and $\sqrt{\mathbf{1-2c}} \notin \mathbf{Q}$:

- **We obtain a family of complete minimal surfaces that correspond to immersions of a sphere punctured at infinitely many points, which depends on a parameter A.**
- **Each surface has infinitely many planar ends**
- **It is not periodic in any variable.**
- **It has infinite total curvature.**

RT wer also applied to the Bonnet family . They are minimal surfaces in R^3 that contain the Enneper surface and the catenoid. (Lemes,_____)

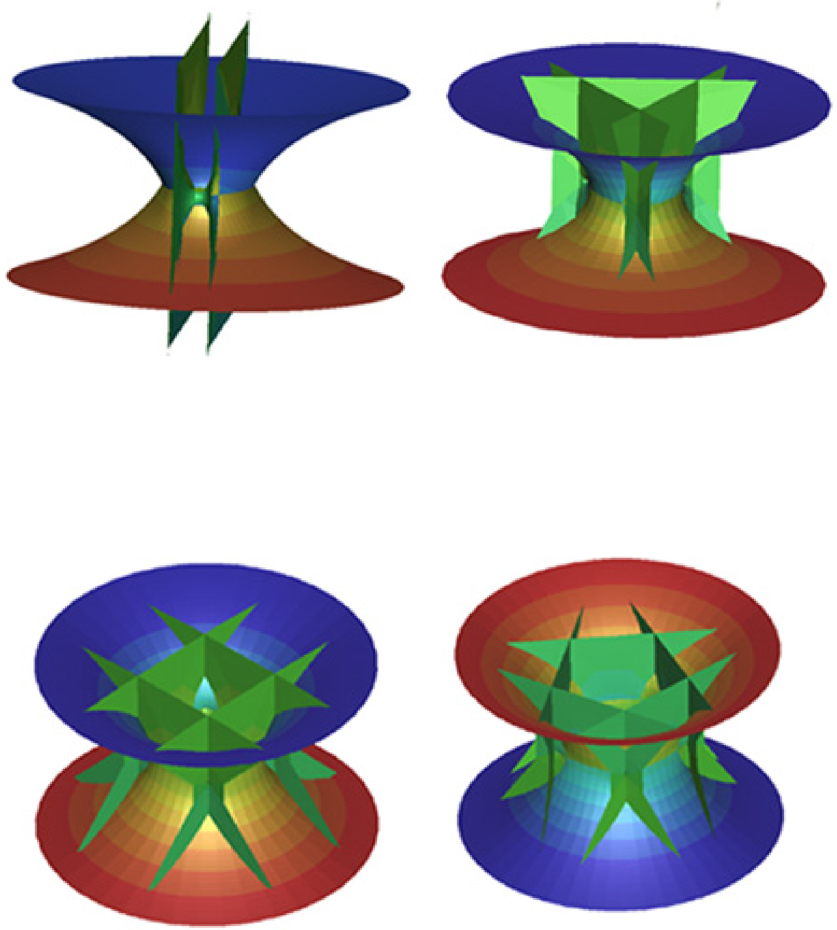


$n = 2,\ m = 1,\ A = 0$

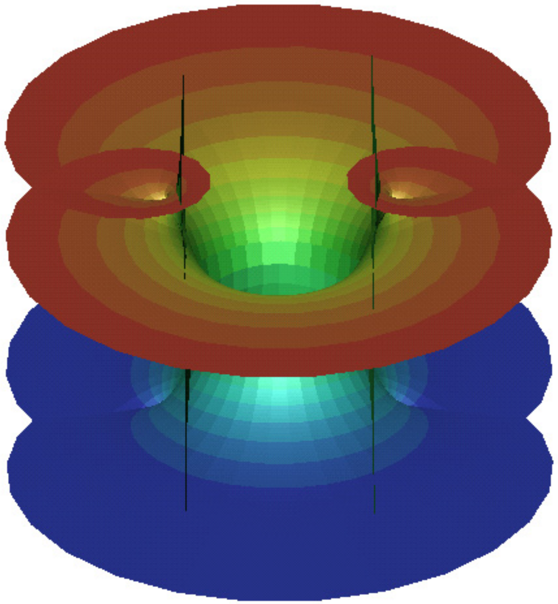


$n = 2,\ m = 1,\ A = 1/2$

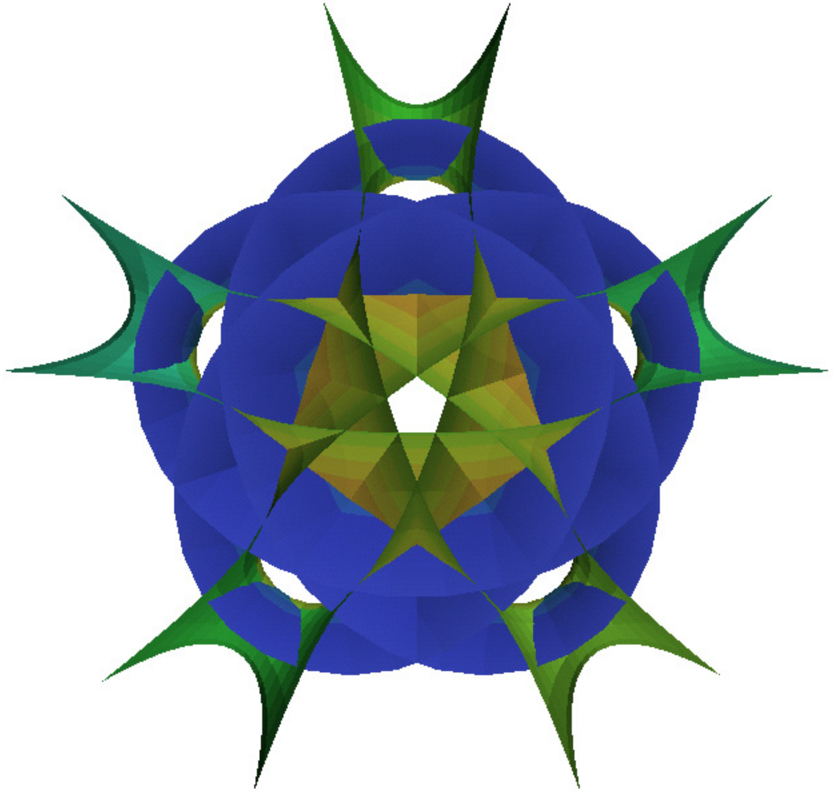
Complete minimal surfaces associated to the catenoid



Complete minimal surf. with $n = 2$, $n = 3$, $n = 4$, $n = 5$ *and* $m = 1$.

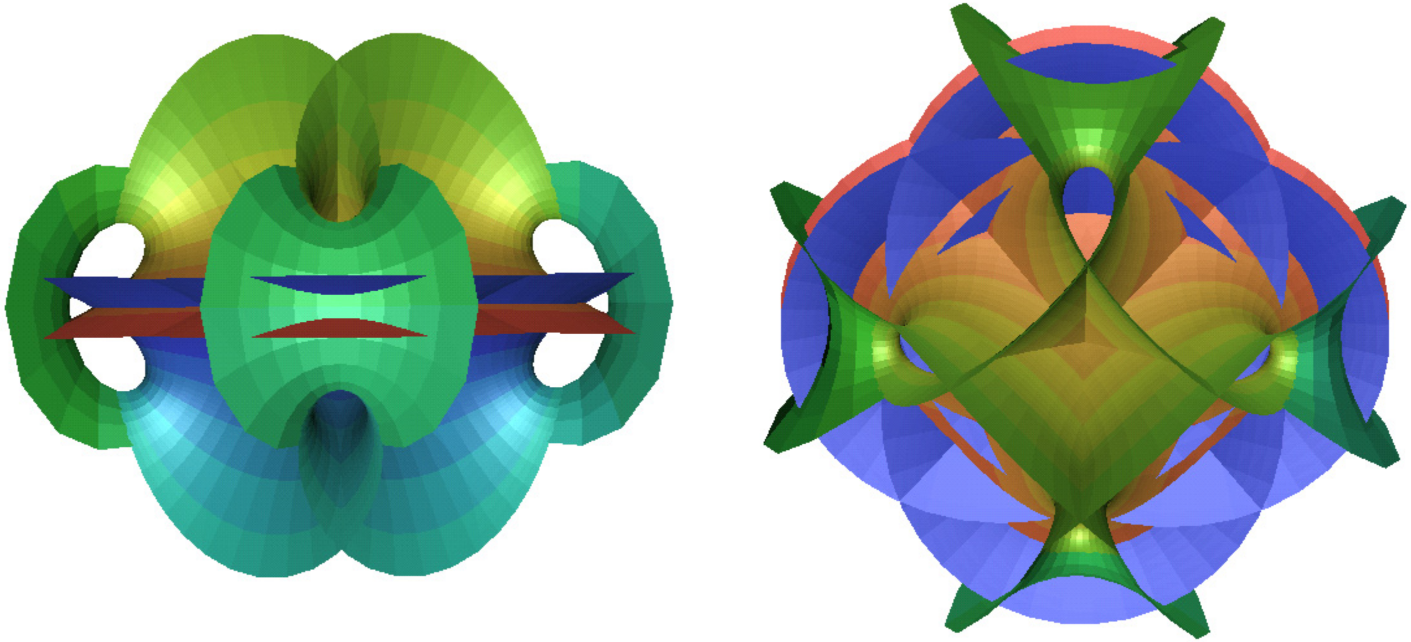


$n = 2,\ m = 3,\ A = 0$



$n = 5,\ m = 3,\ A = 0$

Complete minimal surfaces associated to the catenoid by RT



Complete minimal surface associated to the catenoid $n = 4$ ***and*** $m = 3$

LW surfaces associated to the cylinder

<u>**Proposition**</u> **(Corro, Ferreira, ______) Consider the cylinder**

$$\mathbf{X}(\mathbf{u_1},\mathbf{u_2}) = (\cos(\mathbf{u_2}),\sin(\mathbf{u_2}),\mathbf{u_1})$$

as a LW surface $-1/2+\mathbf{H}+\gamma\mathbf{K}=\mathbf{0}$. The surfaces locally associated to X by a Ribaucour transformation, as in Theorem B, satisfy $-1/2+\tilde{\mathbf{H}}+\gamma\tilde{\mathbf{K}}=\mathbf{0}$ and they are given by

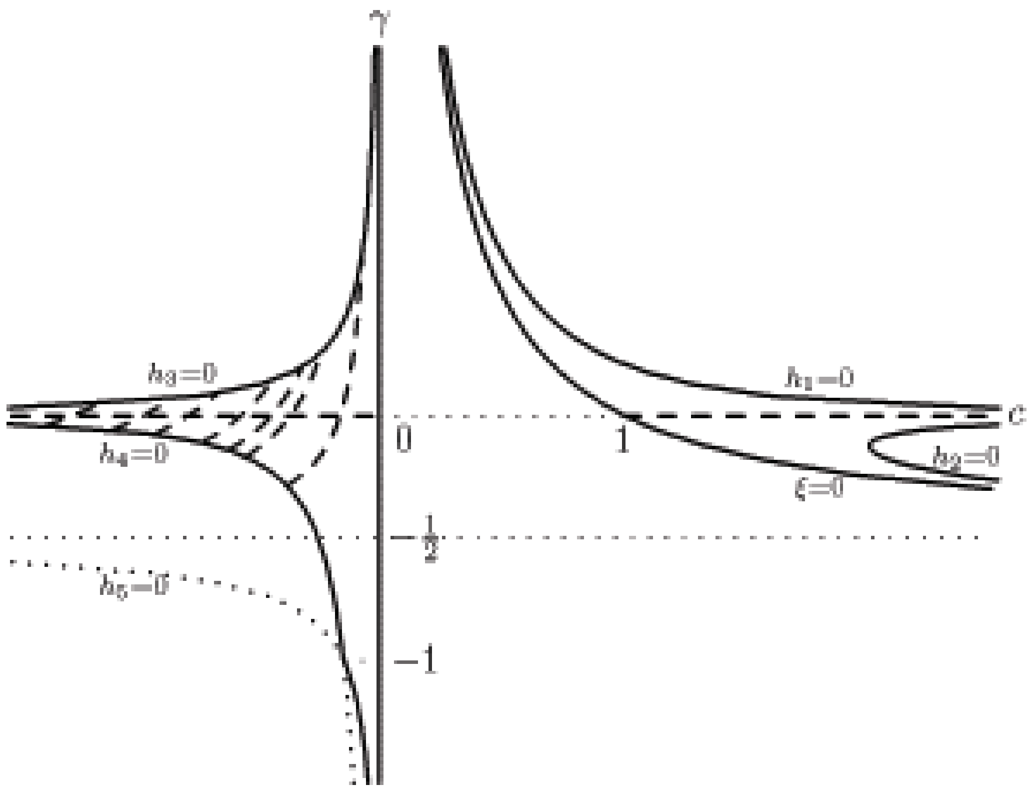
$$\tilde{\mathbf{X}}_{\mathbf{c}\gamma} = \mathbf{X} - \frac{\mathbf{2(f+g)}}{\mathbf{c}[(\mathbf{2}\gamma+\mathbf{1})\mathbf{g}^2-\mathbf{f}^2]}(\mathbf{f}'\mathbf{X}_{\mathbf{u_1}}+\mathbf{g}'\mathbf{X}_{\mathbf{u_2}}-\mathbf{gN})$$

where $\mathbf{c}\neq\mathbf{0}$, $\gamma\in\mathbf{R}$, $\mathbf{f}(\mathbf{u_1})$, $\mathbf{g}(\mathbf{u_2})$ are solutions of

$$\mathbf{f}''+\mathbf{cf}=\mathbf{0}, \qquad\qquad \mathbf{g}''+\xi\mathbf{g}=\mathbf{0}$$

$$\xi(\mathbf{c},\gamma)=\mathbf{1}-\mathbf{c}(\mathbf{2}\gamma+\mathbf{1})$$

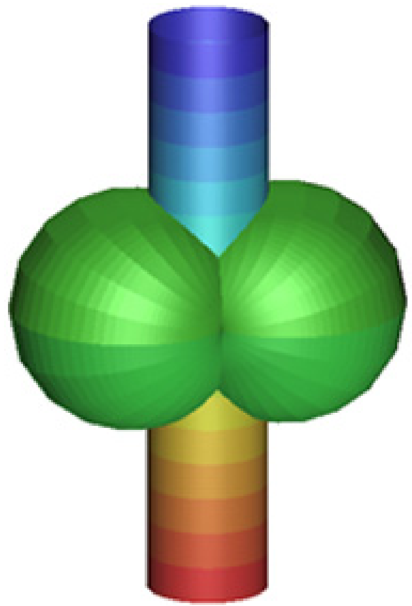
c and ξ are not simultaneously positive.



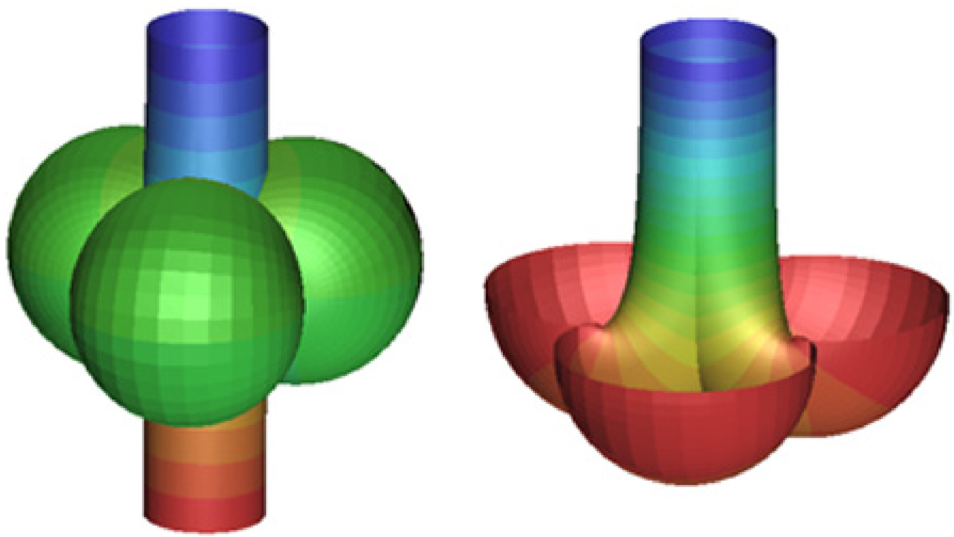
- $\tilde{\mathbf{X}}_{\mathbf{c}\gamma}$ **is complete when** $(\mathbf{c},\gamma)$ **is in one of the two regions.**
- $\tilde{\mathbf{X}}_{\mathbf{c0}}$ **are cmc surfaces (**$\gamma = \mathbf{0}$**).**
- **On the dashed curves on the left** $\xi = \mathbf{1} - \mathbf{c}(\mathbf{2}\gamma + \mathbf{1}) = \frac{\mathbf{n}^2}{\mathbf{m}^2}, \quad \frac{\mathbf{n}}{\mathbf{m}} \in \mathbf{Q}.$

$$\gamma = 0$$

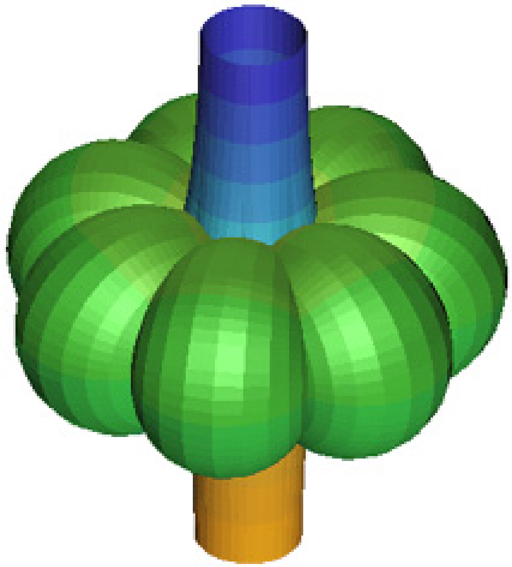
- **For each $c < 0$ such that $\sqrt{1-c} = \frac{n}{m} \in Q$ irreducible, the surface is periodic in one variable.**
- **It has n bubbles and two ends asymptotic to the cylinder with geometric index m.**
- **For other values of c the cmc$H$ surface is not periodic in any variable. The surface has one end and infinitely many bubbles.**

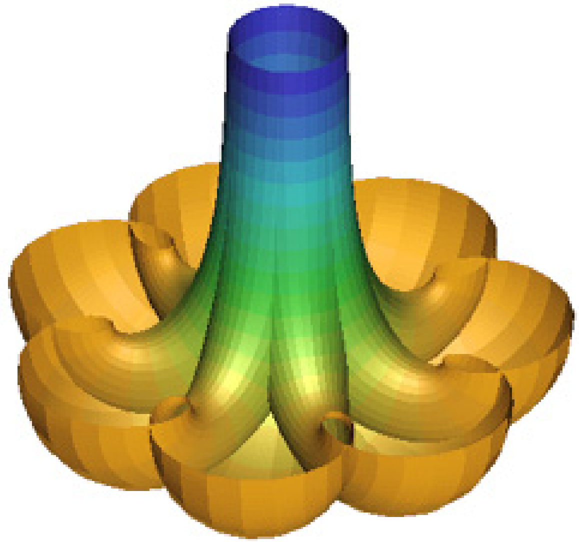


cmc-surface $\gamma = 0$ $\sqrt{1-c} = 2$

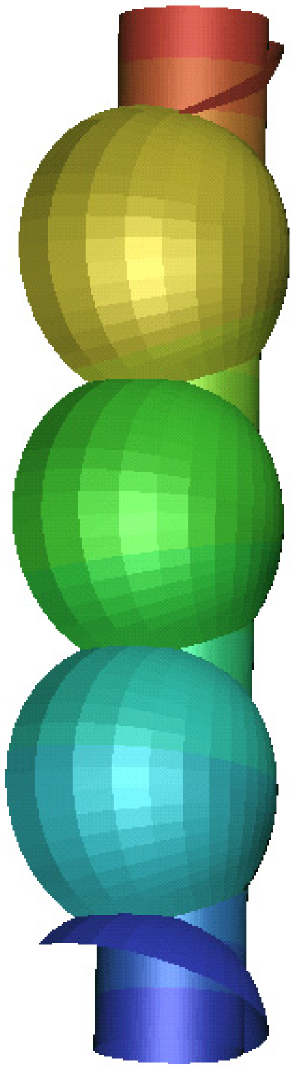


cmc-surface $\gamma = 0$ $\sqrt{1-c} = 3/2$



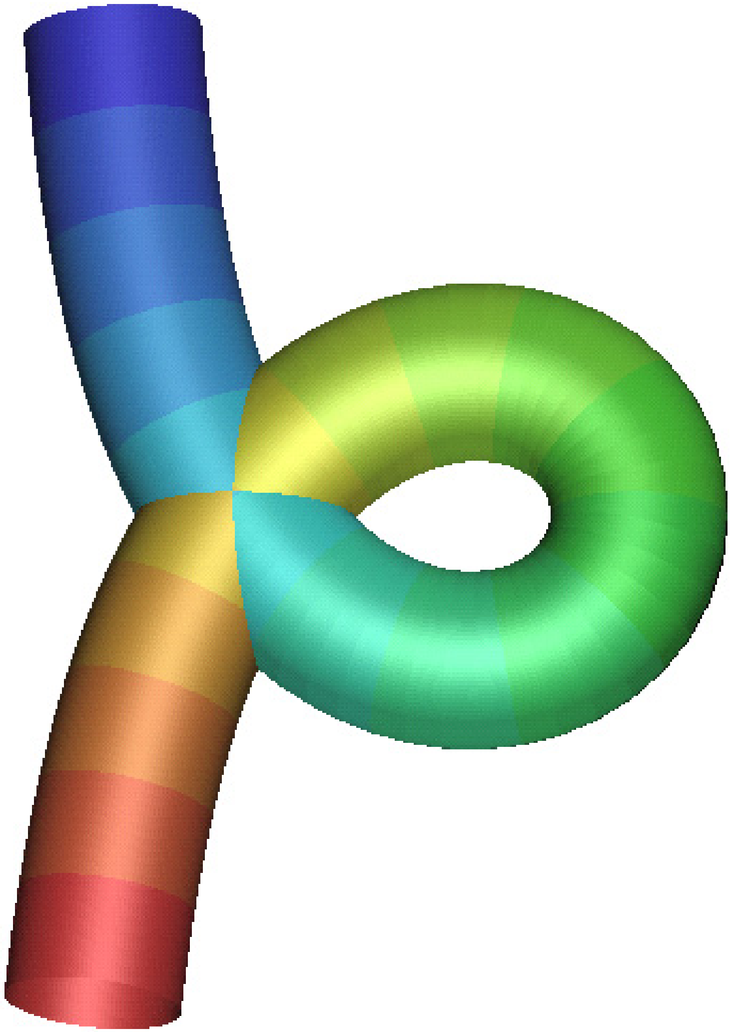


cmc-surface $\sqrt{1-c} = 7/6$

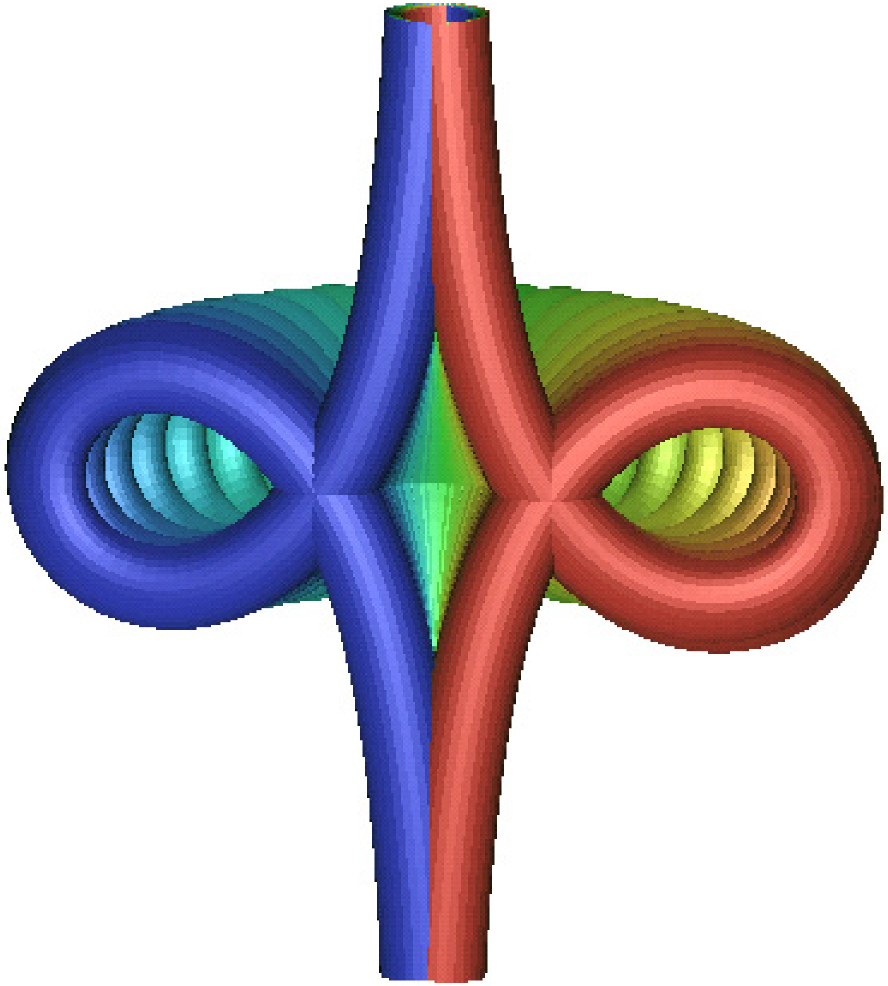


Complete cmc*$1/2$ *surface,* $c = 2.8$*. Infinite number of bubbles

$\gamma \neq 0$

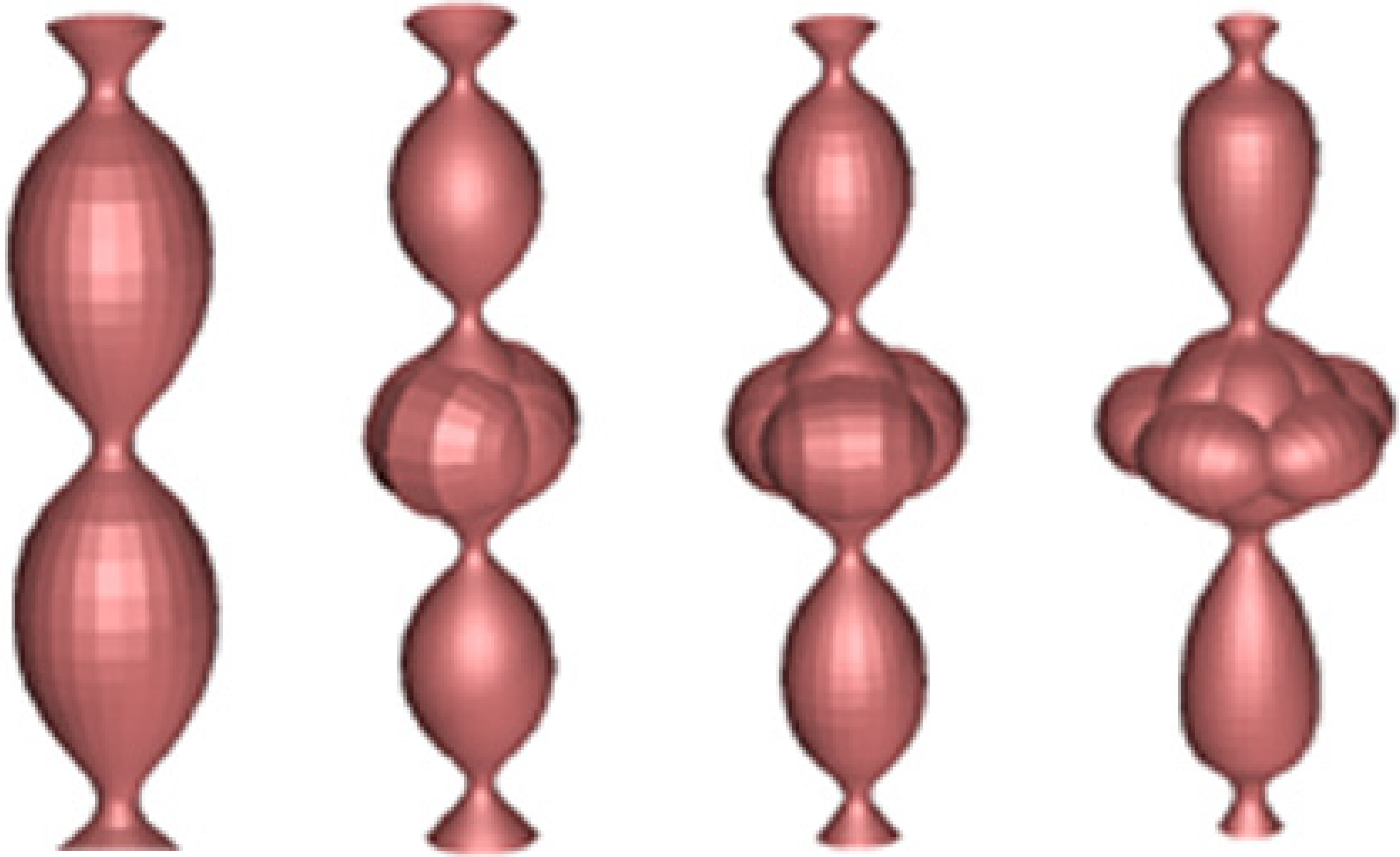


Complete LW surface (tubular) $\gamma = -1/2$, $c = -0.1$.



Partial view of a complete hyperbolic LW surface $\gamma = -1$
Sine-Gordon equation.

Ribaucour transformations of the Delaunay surface



- **DPW method (1998) loop group theory, loop parameter in $\mathbb{C}$.**
- **Burstall (2006): for cmc surfaces in $\mathbb{R}^3$, the simple type dressing for real and pure imaginary parameters are equivalent to Darboux transformations (conformal RT).**
- **Hetrich-Jeromin-Pedit (1997): Bianchi-Backlund transformation of a cmc surface in $\mathbb{R}^3$ is a Darboux transformation. The converse does not hold.**
- **Kobayashi (2008): for a round cylinder in $\mathbb{R}^3$, Bianchi-Backlund $\equiv$ simple type dressing $\equiv$ Darboux transformation.**

- **Kobayashi, using the DPW method, constructed cylinder bubbletons cmc1 in $\mathbb{H}^3$ and cmc0 in S^3. Schmitt used the CMCLab software.**
- **The existence of cmc surfaces with n bubbles was proved by Groβe-Bauckmann 1993 and Sterling-Wente 1993.**
- **The families of cmc surfaces visualized in our papers obtained by using RT, are given by explicit parametrizations.**

M. Lemes, P. Roitman, ____ and R. Tribuzy
Transactions AMS (to appear)

In general, a RT between LW surfaces, given by Theorem B, is not a Darboux transformation, i.e it is not conformal.

<u>Theorem.</u> Let M and $\tilde{M}$ be LW surfaces in $\overline{M}^3(k)$ associated by a Ribaucour transformation as in Theorem B. Then the transformation is conformal

$$\Updownarrow$$

M and $\tilde{M}$ have the same constant mean curvature.

Embedded ends of horosphere type in $\mathbb{H}^3$.

Theorem. **Let** $X : D \subset \mathbb{R}^2 \to \mathbb{H}^3$ **and** $\widetilde{X} : D \setminus \{p_0\} \subset \mathbb{R}^2 \to \mathbb{H}^3$ **be cmc1 surfaces, locally associated by a Ribaucour transformation. Let** $\widetilde{G}$ **and** G **be the hyperbolic Gauss maps of** $\widetilde{X}$ **and** X**, respectively. Assume that** $\Omega_i,\ \Omega$ **and** W **are defined on** D**.**
If $S(p_0) = 0$, $\Omega(p_0) \neq 0$ **and** $S(p) \neq 0$ **for all** $p \in D \setminus \{p_0\}$

$$\Downarrow$$

- $\lim_{p \to p_0} \widetilde{G}(p) = G(p_0)$.
- $\widetilde{X}$ **has an embedded horosphere type end at** p_0**,**

Remarks

- **Mathematicians have been very successful in constructing new complete minimal and constant mean curvature surfaces, by using different techniques.**
- **Lawson correspondence associates isometric surfaces of distinct cmc surfaces in appropriate space forms.**
- **We will relate Lawson correspondence to RT.**

Lawson correspondence

Consider a simply connected cmc$H$ surface $M \subset \overline{M}^3(k)$, with induced metric I and shape operator A.

Let $H' \in \mathbb{R}$, $H' \neq H$. Define $A' := A + (H' - H)Id$.
The pair I, A' satisfies the Gauss and Codazzi equations for a surface $M' \subset \overline{M}'(k')$.
M' is isometric to M, it has constant mean curvature H' and

$$k' = k + H^2 - (H')^2.$$

We say that M and M' are related by the Lawson correspondence

When M is a minimal surface, M' is also referred to as a cmc cousin of M.

Commutativity Theorem

Theorem Lawson correspondence commutes with Darboux transformation (or Ribaucour transformation for surfaces of the same constant mean curvature).

Let M and M' be cmc$H$ and cmcH' surfaces respectively. $H' \neq H$.
$k' + (H')^2 = k + H^2$**.**

$$
\begin{array}{ccccc}
 & & \textit{Lawson} & & \\
 & M \subset \overline{M}^3(k) & \longrightarrow & M' \subset \overline{M}^3(k') & \\
 & & & & \\
Ribaucour(c) & \downarrow & & \downarrow & Ribaucour(c') \\
 & & & & \\
 & \tilde{M} \subset \overline{M}^3(k) & \longrightarrow & \tilde{M}' \subset \overline{M}^3(k') & \\
 & & \textit{Lawson} & &
\end{array}
$$

$c \neq 0$**,** $c' \neq 0$**.**

Verify commutativity considering

$$c' = c + H' - H \qquad \Omega' = \Omega, \qquad W' = W + (H' - H)\Omega.$$

Corollary. **Let** $X : U \subset \mathbb{R}^2 \to \overline{M}^3(k)$ **and** $X' : U \to \overline{M}^3(k')$ **be immersions related by the Lawson correspondence,** U **simply connected.**

Let $\widetilde{X}$ **and** $\widetilde{X}'$ **be the Ribaucour transformations of** X **and** X'**, with constants** c **and** c' **resp.**

$$\Downarrow$$

- **The surfaces** $\widetilde{X}$ **and** $\widetilde{X}'$ **are defined on the same subset of** U**.**
- **The surfaces of the family** $\widetilde{X}$ **are complete if, and only if the surfaces of** $\widetilde{X}'$ **are complete.**

Applications to the cousins of the catenoid

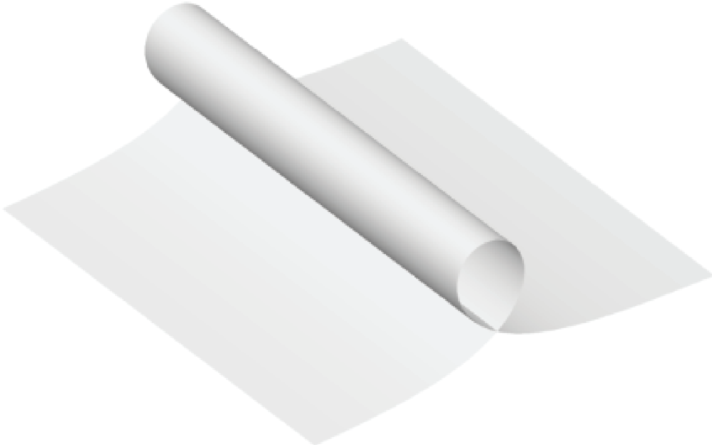
Consider the family of homothetic catenoids in $\mathbb{R}^3$ parametrized by

$$X(u_1,u_2) = \frac{\gamma}{2}(\cos 2u_2 \cosh 2u_1,\ \sin 2u_2 \cosh 2u_1,\ -2u_1).$$

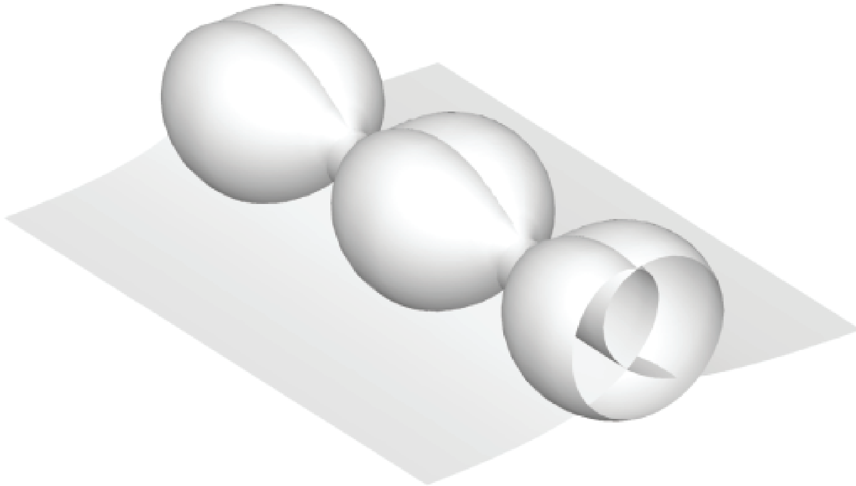
where $(u_1,u_2) \in \mathbb{R}^2$.

Solve the Ribaucour transformation for this family and apply to each cousin of the catenoid.

The so called singular catenoid cousin is the cousin of the catenoid where $\gamma = 1$.



Singular catenoid cousin



Complete cmc1 surface

The associated cmc1 surface, by RT with $c = -3$***, has an embedded horosphere type end at each pair*** $(0, u_2^0)$***, where*** $u_2^0 = (n - \frac{1}{4})\frac{\pi}{2}$ ***and*** n ***is an integer.***

Application:
The Bonnet family of minimal surfaces in $\mathbb{R}^3$ and cousins in $\mathbb{H}^3$

We consider the Bonnet family described by the Weierstrass data

$$g(z) = e^{\mu z}, \qquad f(z) = 2\frac{\nu}{\mu}e^{-\mu z}, \qquad \nu \in \mathbb{R},\ \mu \in \mathbb{C}$$

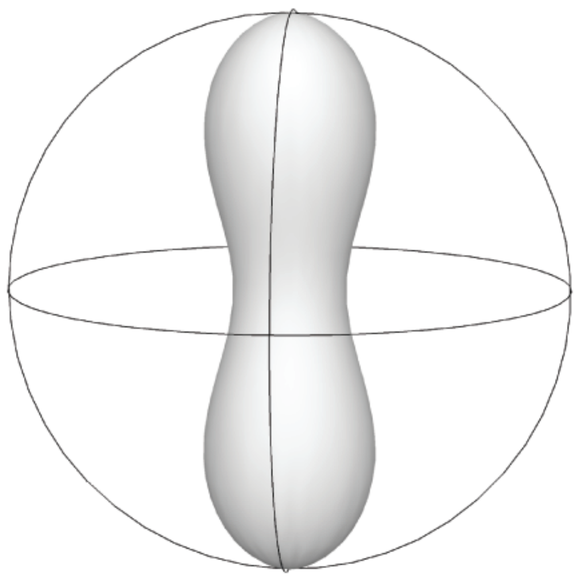
The cousin surfaces differ according to the values of ν.

- **For a parametrization $X(z,\bar{z})$ of the cmc1 cousin surface in $\mathbb{H}^3$ we get explicit parametrizations for the associated surfaces, using the proof of the Commutativity Theorem.**

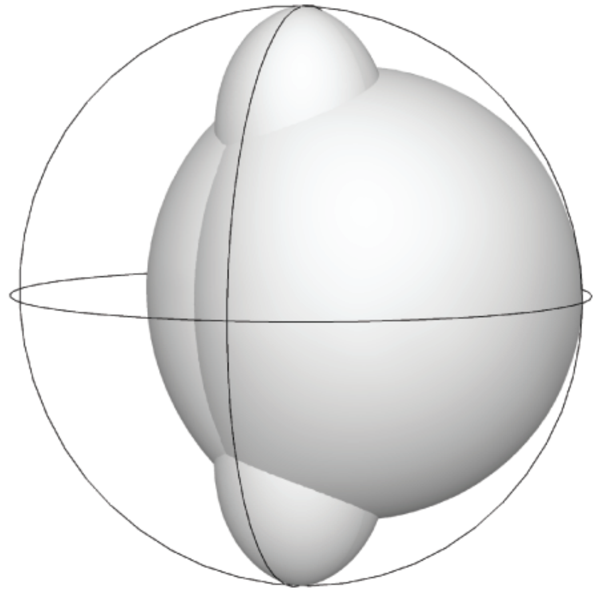
Example: a catenoid cousin and the associated surfaces

The associated surfaces depend on two parameters c, A.

A nonsingular catenoid cousin: $\mu = 2 \qquad \nu = 5/4$.



A catenoid cousin



cmc1 surface c=4/5

A cmc1 cousin of the catenoid and an associated complete surface by Ribaucour transformation, with $c = 4/5, A = 0$,. It has 2 ends, one of them is an embedded horosphere type end.

Special class of associated surfaces

Consider $c \in \mathbb{R} \setminus \{0, -1\}$**.**

If $c < \frac{4}{5}$ **and** $c = \frac{1}{5}(4 - 9\frac{n^2}{m^2})$ **and** $\frac{n}{m} \in \mathbb{Q}$ **is irreducible,**

$$\Downarrow$$

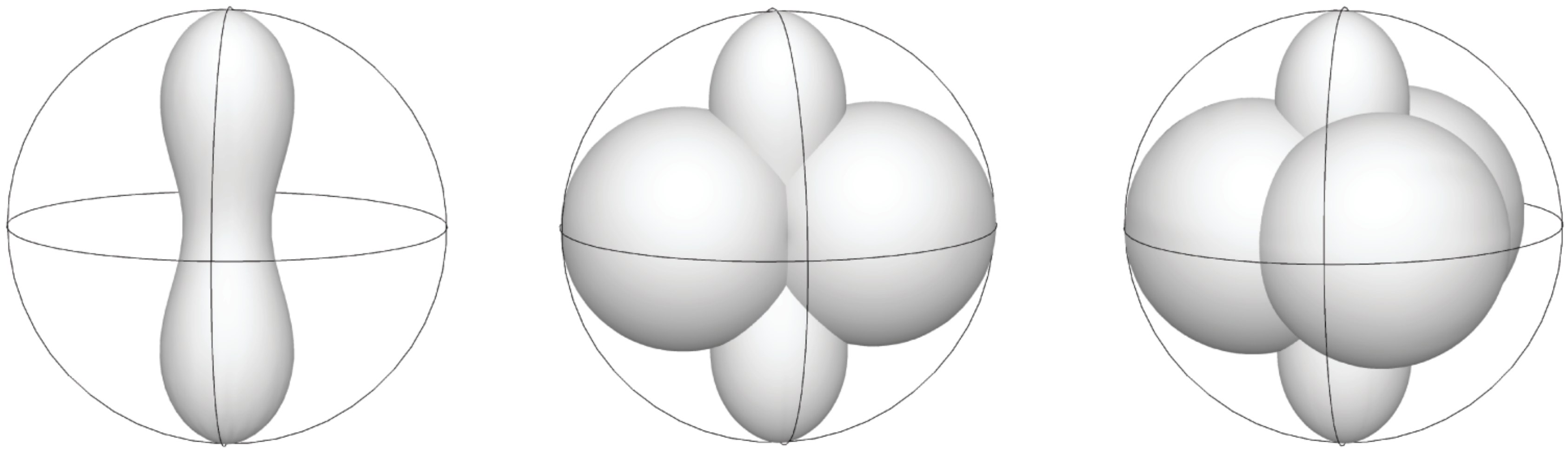
- **The associated cmc**1 **surface is periodic in** u_2
- n **is the number of embedded ends of horosphere type**
- m **is the geometric index of the end of catenoid type**
- **It is the immersion of a sphere punctured at** $n+2$ **points**
- **The total curvature is** $-4\pi(n+m)$

Other values of c

If $c > \frac{4}{5}$ **or** $c < \frac{4}{5}$ **and** $\frac{1}{3}\sqrt{4-5c} \notin Q$

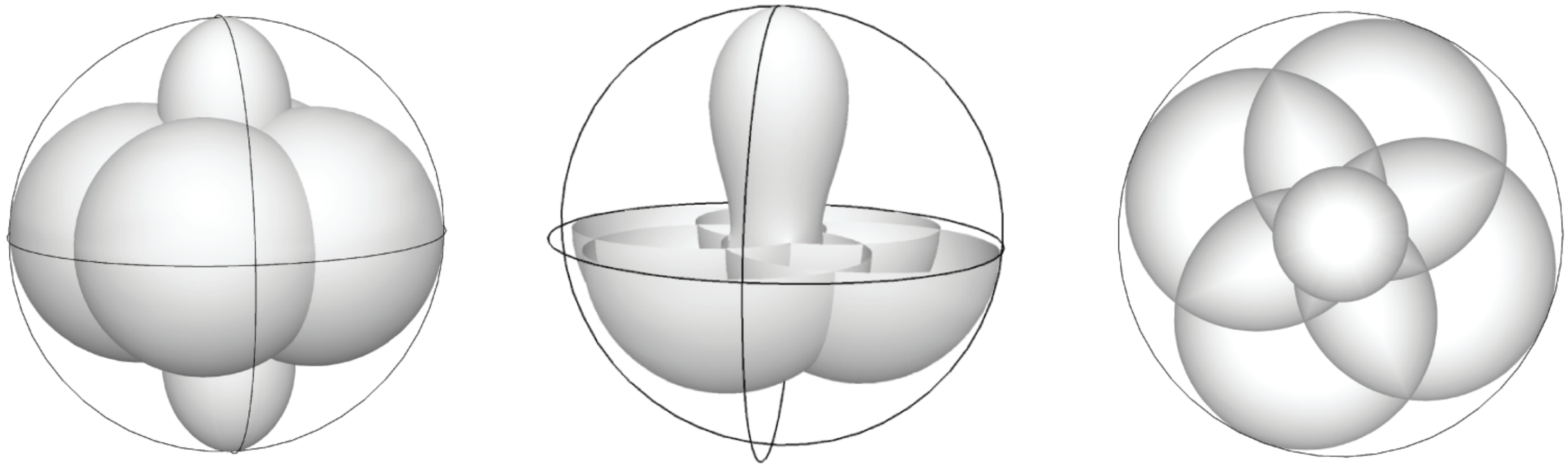
$$\Downarrow$$

- **the associated cmc**1 **surfaces are not periodic in any variable**
- **it has infinitely many embedded ends of horosphere type.**



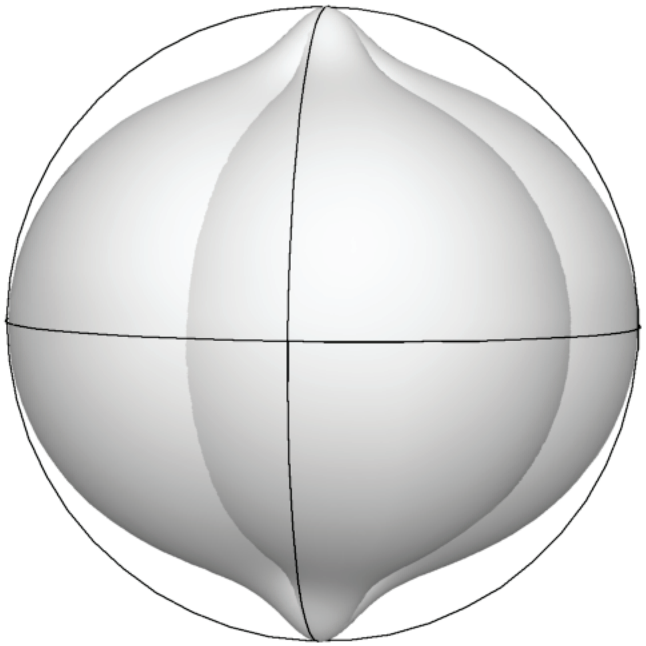
cmc1 surface $n = 2,\ m = 1$ $n = 3,\ m = 1$

A cmc1 cousin of the catenoid in $\mathbb{H}^3$ *and associated complete surfaces by Ribaucour transformations with* $A = 0$.

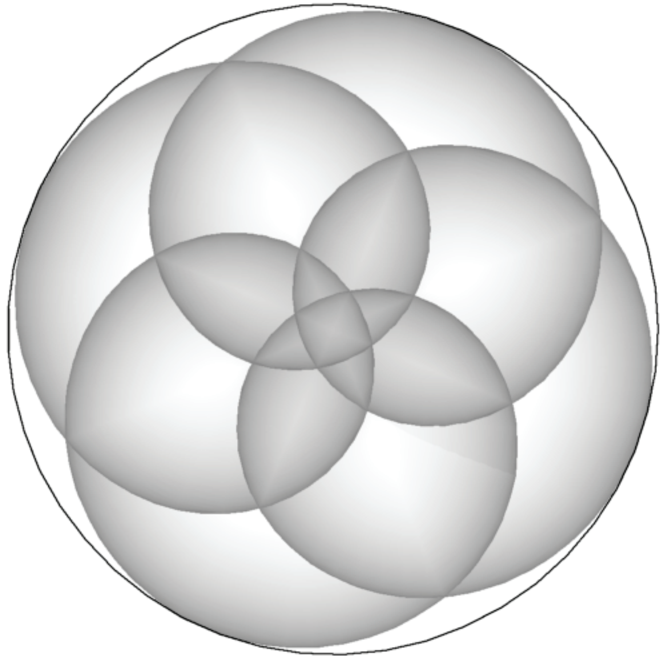


$n = 4,\ m = 1$ *Half surface* *Top view, half surface*

A complete cmc1 surface in $\mathbb{H}^3$ *associated to the catenoid cousin.*



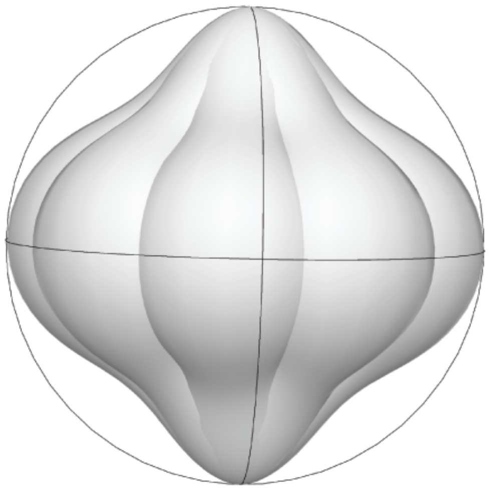
$n = 4,\ m = 5$



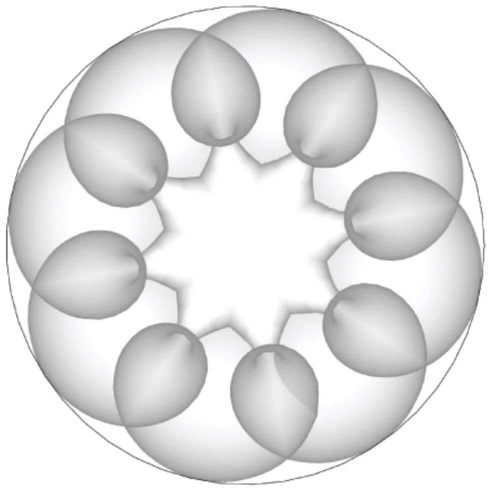
Top view of half surface

A complete cmc1 surface reflected to the upper half space (or internal component of the Poincaré ball model) of $\mathbb{H}^3$.

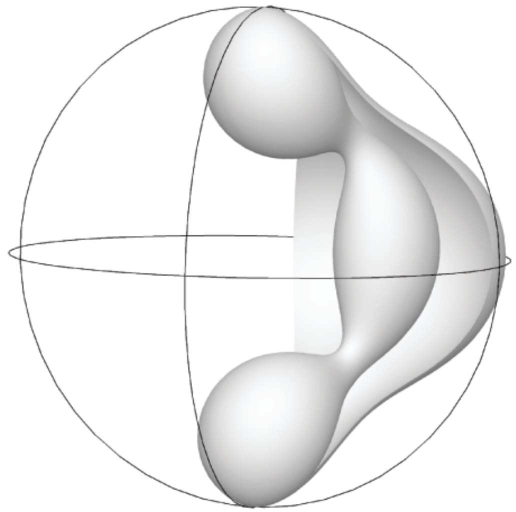
It has 4 embedded ends of horosphere type and 2 ends of catenoid type of geometric index 5.



$n = 8, m = 9$



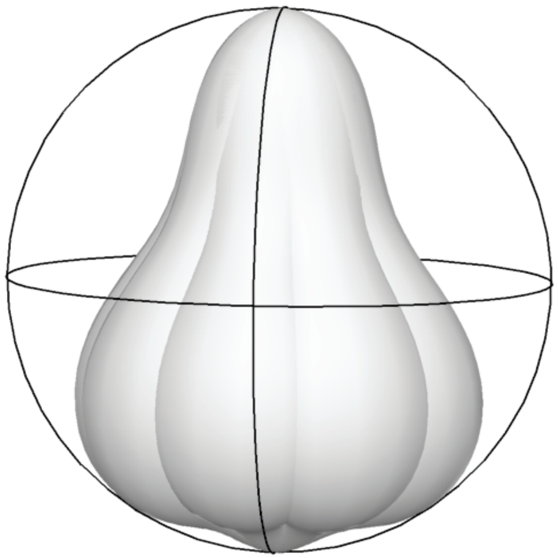
Top view, half surface



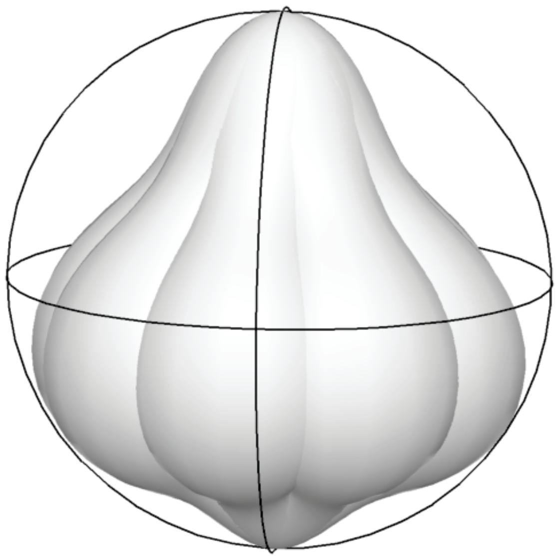
Part of inner view

A complete cmc1 surface reflected to the upper half space (or internal component of the Poincaré ball model) of $\mathbb{H}^3$.
It has 8 embedded ends of horosphere type and two ends of catenoid type of geometric index 9.

$A \neq 0$



n=8, m=9, A=-1

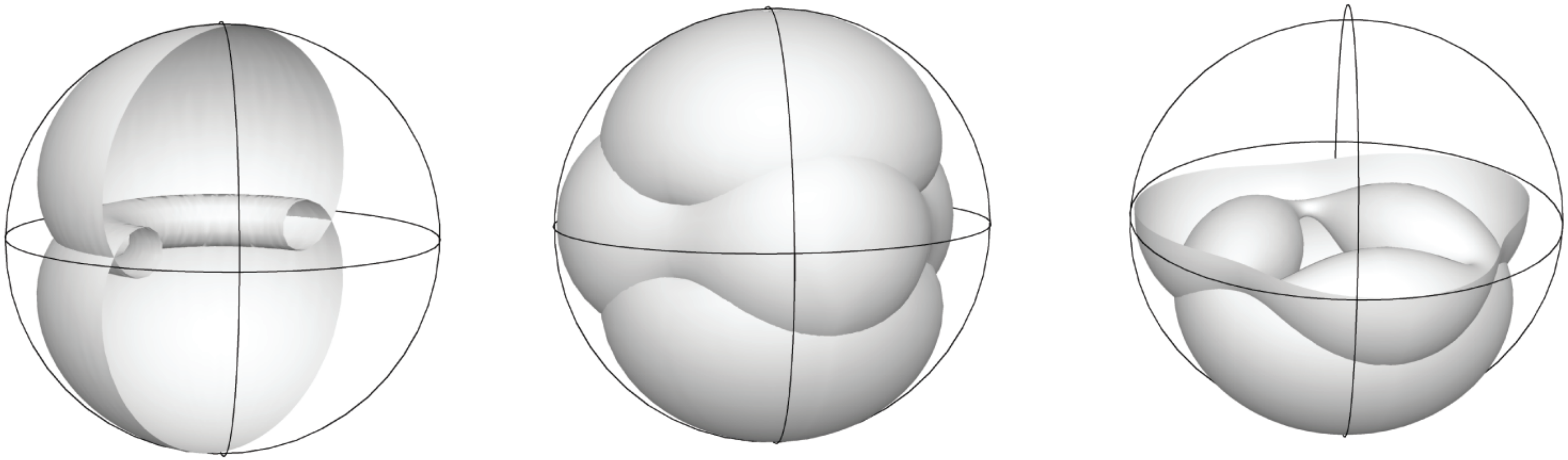


n=8, m=9, A=-1/2

Two complete cmc1 surfaces reflected to the upper half space of $\mathbb{H}^3$. They have 8 embedded ends of horosphere type and two catenoid ends with geometric index 9.

Umehara Yamada examples

$\mu = 5/2$, $\nu = (1-\mu^2)/4$, $A = 0$ **and specific values for** c**.**



Half of a cmc1 non embedded cousin of the catenoid in $\mathbb{H}^3$*. A complete cmc1 surface associated by a RT with* $n = 3$, $m = 1$ *and* $A = 0$*. It has three embedded ends of horosphere type and two embedded ends of catenoid type.*

- **Other values for** c **or** A **produce new cmc1 surfaces.**

Theorem

- **Each minimal surface in $\mathbb{R}^3$ associated by Ribaucour transformation to the Bonnet family is complete.**
- **Each cmc1 surface in $\mathbb{H}^3$ associated by a Ribaucour transformation to the cousins of the Bonnet family is complete.**

Application: A family of cmc $H = -\sqrt{5}/2$ surfaces in $\mathbb{H}^3$

Start with the cylinder in $\mathbb{R}^3$ with radius one.

$$X(u_1, u_2) = (\cos u_2,\ \sin u_2,\ u_1).$$

Surface in $\mathbb{H}^3 \subset L^4$ corresponding to X by Lawson correspondence

$$X'(u_1, u_2) = (\frac{2}{\alpha}\cosh\frac{\alpha u_2}{2}, \frac{1}{\beta}\cos(\beta u_1),\ \frac{1}{\beta}\sin(\beta u_1),\ \frac{2}{\alpha}\sinh\frac{\alpha u_2}{2}),$$

where

$$\alpha = \sqrt{2(\sqrt{5}-1)}, \qquad \beta = \frac{\alpha}{\sqrt{5}-1}.$$

X' **has constant mean curvature** $H = -\sqrt{5}/2$**.**

Special class of associated cmc$H$ surfaces

Consider $c<0$ **or** $c>1$ **and** $c \neq (\sqrt(5)+1)/2$**.**

If $c=\frac{(\sqrt{5}+1)n^2}{2m^2}$**,** $\frac{n}{m} \in Q$ **is irreducible and** $\begin{cases} \frac{n}{m}>1, \text{ or} \\ (\sqrt{5}-1)/2<n^2/m^2<1 \end{cases}$

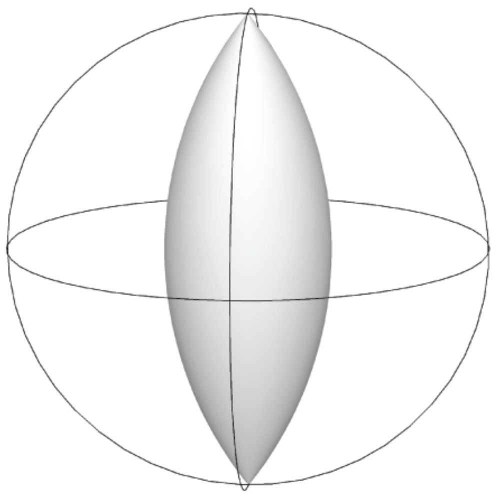
$$\Downarrow$$

- **the associated cmc$H$ surface is periodic in u_1**
- n **is the number of bubbles or "sections"**
- m **is the geometric index of the two ends**

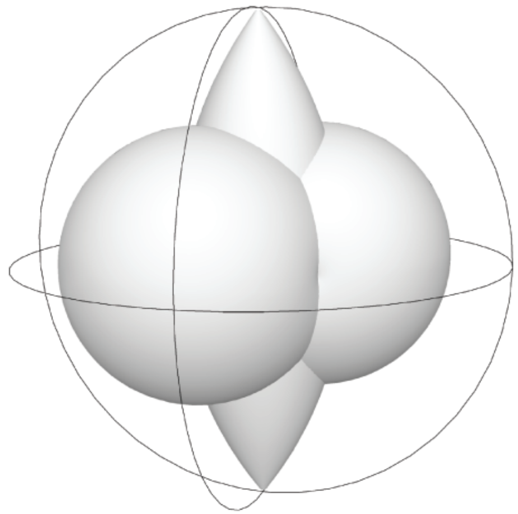
Otherwise: $c<0$ **or** $c>1$**,** $c \neq (\sqrt(5)+1)/2$ **and** $\sqrt{\frac{2c}{\sqrt{5}+1}} \notin Q$

$$\Downarrow$$

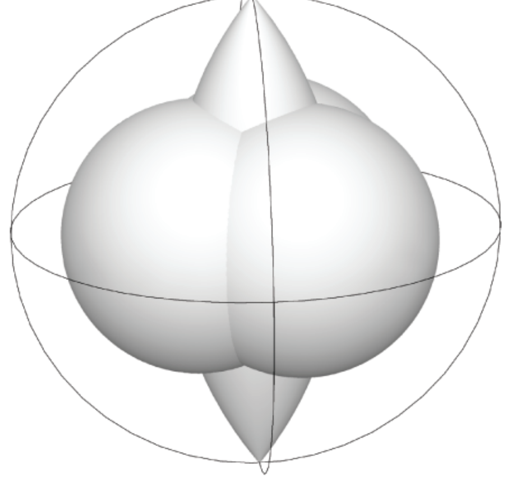
- **the associated cmc$H$ surfaces are not periodic in any variable**
- **it has infinitely many bubbles or "sections".**



A cylinder in $\mathbb{H}^3$

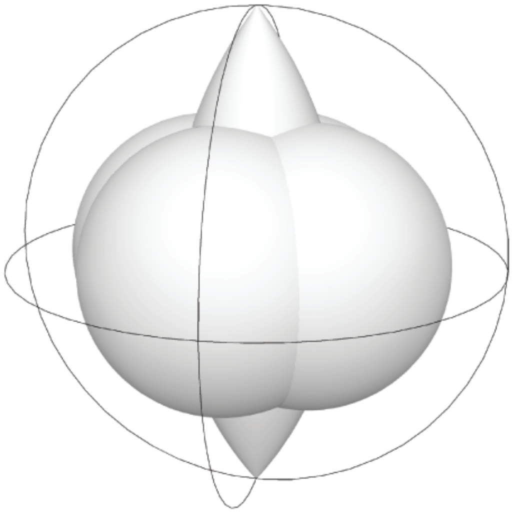


$n = 2, \quad m = 1$

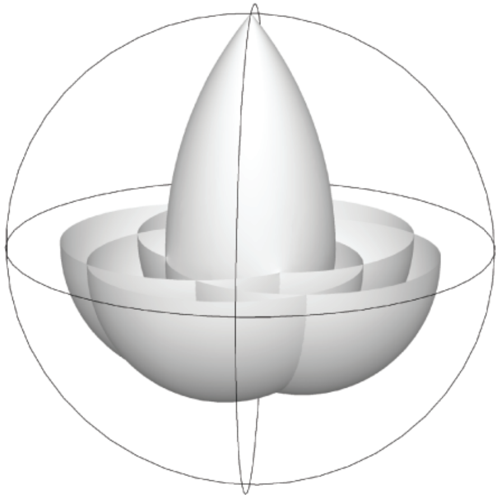


$n = 3, \quad m = 1$

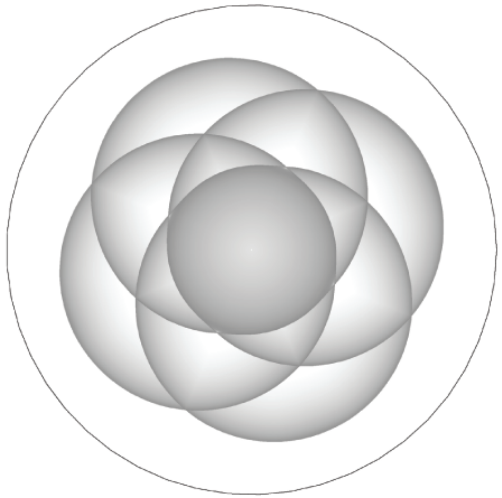
A cylinder in $\mathbb{H}^3$ *with constant mean curvature* $H = -\sqrt{5}/2$. *Two complete cmcH surfaces associated to the cylinder by Ribaucour transformations with 2 and 3 "bubbles" and 2 embedded ends of cylindrical type.*



n=4,   m=1

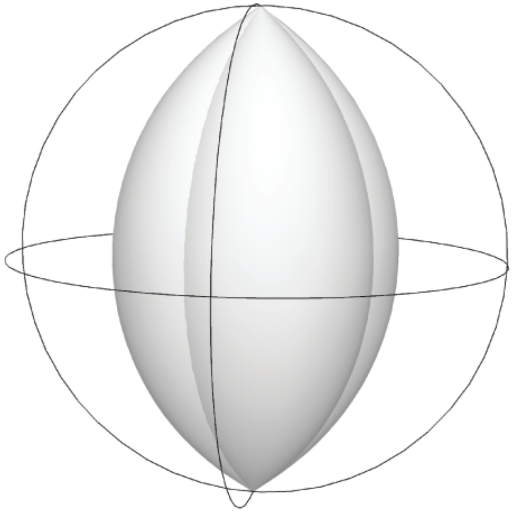


Half surface

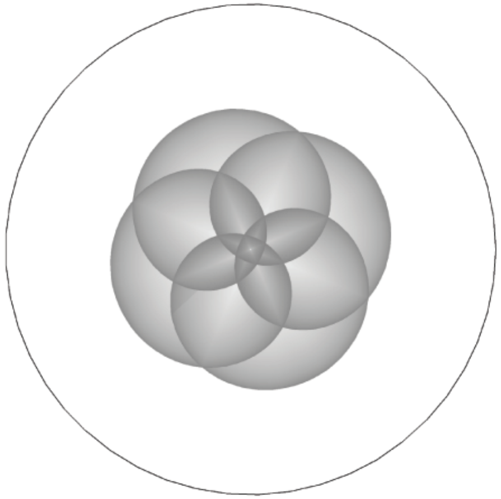


Top view, half surface

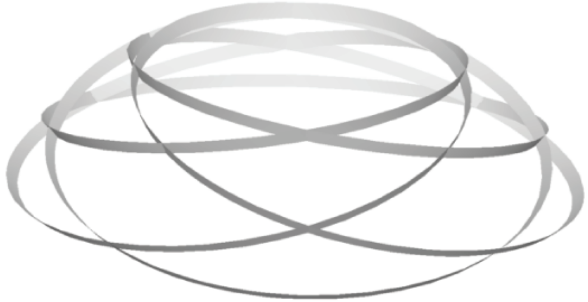
A complete cmcH surface in $\mathbb{H}^3$ It has four bubbles and two embedded ends of cylindrical type.



n=4, m=5

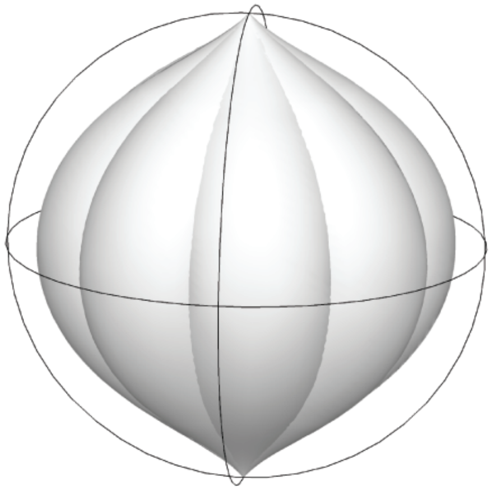


Top view of lower half

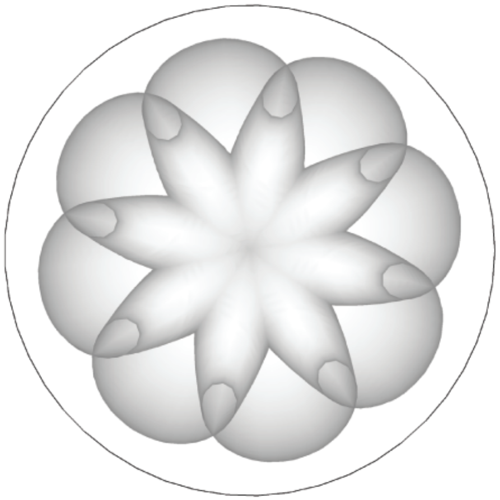


Close to an end

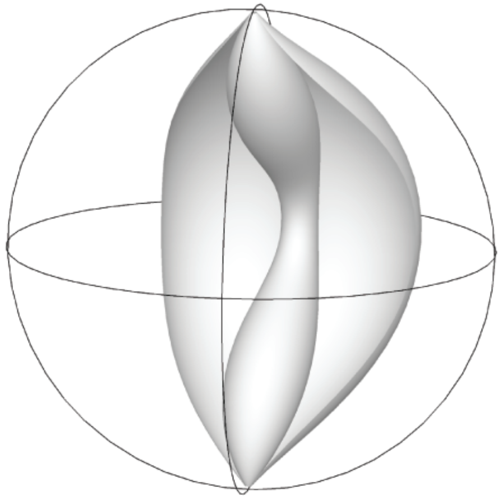
A complete cmcH surface reflected to the upper half space of $\mathbb{H}^3$ *(or the inner part of the Poincaré ball) It has 4 “sections” and two cylindrical ends of geometric index 5.*



n=8   m=9

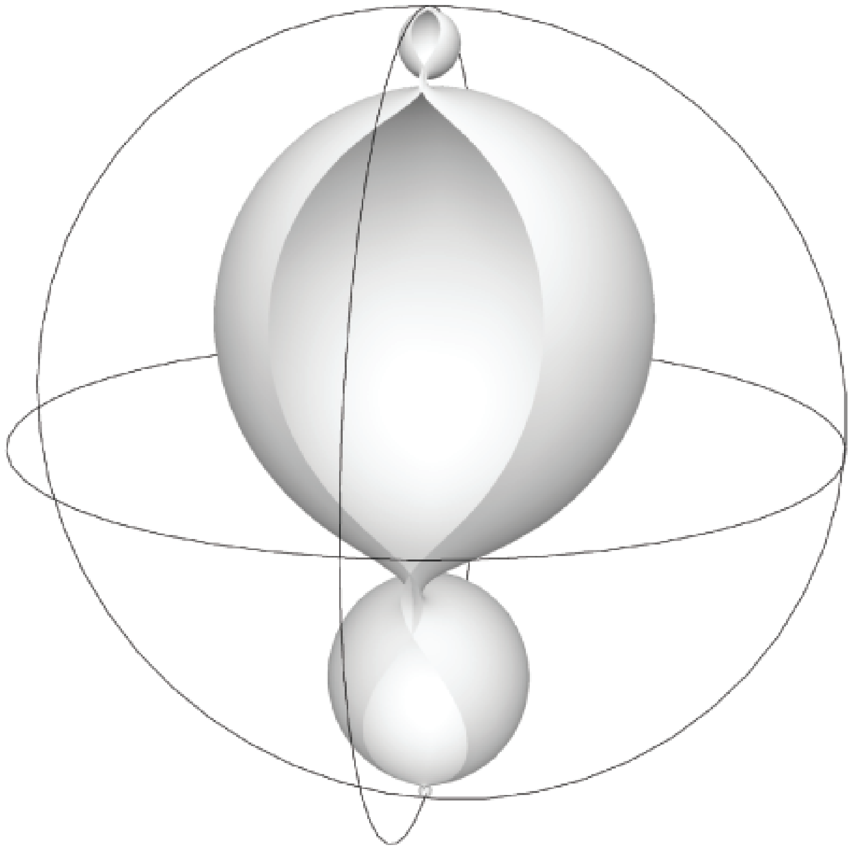


Top view of lower half



Part of inner view

A complete cmcH surface reflected to the upper half space of $\mathbb{H}^3$ (or the inner part of the Poincaré ball) It has 8 "sections" and two cylindrical ends with geometric index 9.



Part of a complete cmcH surface in $\mathbb{H}^3$, obtained with $c = -3/2$. It has infinitely many bubbles in both directions approaching the boundary of the Poincaré ball model.