Weyl metrisability for projective surfaces

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Projective structure

M, connected, smooth n-manifold, $n \geq 2$. ∇ affine connection with $T_\nabla = 0$.

∇_1, ∇_2 are said to be projectively equivalent if they have the same geodesics, considered as unparametrised curves. An equivalence class $[\nabla]$ is called a projective structure.

Projectively equivalent connections induce the same parallel transport on the projectivised tangent bundle $\mathbb{P}(TM) = (TM \setminus \{0_M\})/\mathbb{R}^*$

Theorem (Weyl, 1921)**.** *∇^1, ∇^2 are projectively equivalent iff there exists a 1-form β such that*

$$\nabla^1_X Y - \nabla^2_X Y = \beta(X)Y + \beta(Y)X.$$

$(M^n, [\nabla])$ is called projectively flat if locally the geodesics can be mapped to straight lines in $\mathbb{R}^n$.

Weyl projective curvature

Theorem (Thomas, 1925)**.** *For a projective structure* $[\nabla]$, *the functions (defined w.r.t coordinates)*

$$\Pi^i_{kl} = \Gamma^i_{kl} - \frac{1}{n+1}\left(\delta^i_k \Gamma^a_{al} + \delta^i_l \Gamma^a_{ak}\right)$$

are projectively invariant and locally fully describe $[\nabla]$.

Theorem (Weyl, 1921)**.** *A projective n-manifold* $(M, [\nabla])$, $n \geq 3$, *is projectively flat iff the Weyl projective curvature tensor*

$$W^i_{jkl} = B^i_{jkl} + \frac{1}{n-1}\left(\delta^i_k B^a_{jla} - \delta^i_l B^a_{jka}\right),$$

$$B^i_{jkl} = \Pi^i_{lj,k} - \Pi^i_{kj,l} + \Pi^a_{lj}\Pi^i_{ka} - \Pi^a_{kj}\Pi^i_{la},$$

vanishes. For $n = 2$, $W^i_{jkl} = 0$, *but there are other obstructions to flatness.*

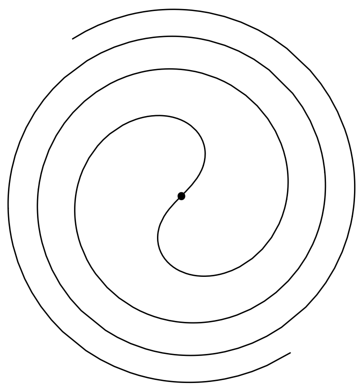
Problem (R. Liouville, 1889). *Given* $(M^2, [\nabla])$ *does there exist a Riemannian metric* g *with* $D^g \sim \nabla$*?*

Theorem (R. Liouville 1889). $D^g \sim \nabla$ *iff*

$$\mathrm{d}_{[\nabla]}\left(g \otimes \det g^{-2/3}\right) = 0$$

where $\mathrm{d}_{[\nabla]}$ *is a linear first order differential operator which is projectively invariant.*

Example $D^{g_E} + g_E \otimes \beta^{\#}$ with $\beta = -y\mathrm{d}x + x\mathrm{d}y$ is not projectively equivalent to a Levi-Civita connection.



Bryant, Dunajski & Eastwood solve Riemannian metrisability problem for real analytic projective surfaces in 2009.

Related problems

Theorem (Alvarez-Paiva & Berck, 2010, Bryant). *Locally every surface path geometry is Finsler metrisable.*

An affine torsion-free connection preserving a conformal structure is called a Weyl connection.

Theorem (Weyl, 1918). ∇ *preserves* $[g]$ *iff* $\exists$ *a* 1*-form* β *such that*

$$\nabla g = 2\beta \otimes g. \tag{1}$$

Given (g, β), $D^{g,\beta} : (X, Y) \mapsto D^g_X Y + g(X, Y)\beta^\# - \beta(X)Y - \beta(Y)X$ *is the unique connection solving (1).*

$(e^{2f} g, \beta + \mathrm{d}f)$ induces the same Weyl connection. Equivalence class $[g, \beta]$ is called a Weyl structure.

$[g, \beta]$ is called compatible with $[\nabla]$ if $D^{g,\beta} \sim \nabla$.

Problem. *Given* $(M^2, [\nabla])$, *does there exist a Weyl structure compatible with* $[\nabla]$? *(*β *exact* $\longleftrightarrow$ *Riem. metrisability problem)*

Theorem (Cartan, 1921)**.** *Given* $(M^2, [\nabla])$*, there exists a* $\mathrm{SL}(3,\mathbb{R}) \supset H$*-bundle* $\pi : B \to M$ *and* $\theta \in \Omega^1(B, \mathfrak{sl}(3,\mathbb{R}))$ *such that*

(i) $\theta_b : T_bB \to \mathfrak{sl}(3,\mathbb{R})$ *is an isomorphism and* $R_h^*\theta = h^{-1}\theta h$*.*

(ii) $\theta(X_v) = v$ *for every fundamental vector field of* $\pi : B \to M$*.*

(iii) *Writing* $\theta = (\theta^i_k)_{i,k=0..2}$*, the leaves of the foliation given by* $\left(\mathbb{R}\left\{\theta^2_0, \theta^2_1\right\}\right)^{\perp}$ *project to geodesics on M.*

(iv) *For* $\omega \in \Omega^2(M)$ *with* $\omega > 0$*,* $\pi^*\omega = f\theta^1_0 \wedge \theta^2_0$ *for* $f \in C^\infty(B, \mathbb{R}^+)$*.*

(v) *The curvature 2-form* $\Theta = \mathrm{d}\theta + \theta \wedge \theta$ *satisfies*

$$\Theta = \begin{pmatrix} 0 & L_1\,\theta^1_0 \wedge \theta^2_0 & L_2\,\theta^1_0 \wedge \theta^2_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some smooth functions $L_i : B \to \mathbb{R}$*.*

Curvature $\left(L_1\theta^1_0 + L_2\theta^2_0\right) \otimes \left(\theta^1_0 \wedge \theta^2_0\right) = -\star\mathrm{d}K \otimes \mathrm{d}A$ *for* $\nabla \sim D^g$*.*

B consists of the 2-jets of local diffeomorphisms $\varphi : B_\varepsilon 0 \to M$ which are adapted to the orientation and projective structure on M.

$\varphi_0' : T_0\mathbb{R}^2 \to T_{\varphi(0)}M$ is orientation preserving and $x = \varphi^{-1} \circ \gamma$ has vanishing curvature at $0 \in \mathbb{R}^2$ for every $[\nabla]$-geodesic γ with $\gamma(0) = \varphi(0)$.

H is the Lie group of 2-jets of linear fractional transformations of the form

$$f_{a,b} : x \mapsto \frac{a \cdot x}{1 + b \cdot x}, \quad a \in \mathrm{GL}^+(2, \mathbb{R}), b^t \in \mathbb{R}.$$

$$\tilde{H} = \left\{ h_{a,b} = \begin{pmatrix} \det a^{-1} & b \\ 0 & a \end{pmatrix} \middle| a \in \mathrm{GL}^+(2, \mathbb{R}), b^t \in \mathbb{R}^2 \right\}$$

$h_{a,b} \mapsto j_0^2 \left(f_{\tilde{a},\tilde{b}} \right)$ where $\tilde{a} = a \det a, \tilde{b} = b \det a$ identifies H with $\tilde{H} \subset \mathrm{SL}(3, \mathbb{R}) \simeq \mathrm{PL}(2, \mathbb{R})$.

Coordinate section

Orientation preserving coordinates $x : U \to \mathbb{R}^2$ induce a coordinate section sending $p \in U$ to the 2-jet $j^2_0\varphi$ defined by

$$\varphi(0) = p, \quad \partial_k (x \circ \varphi)^i(0) = \delta^i_k, \quad \partial_k \partial_l (x \circ \varphi)^i(0) = -\Pi^i_{kl}(p).$$

Coordinate sections satisfy

$$\sigma_x^* \theta = \begin{pmatrix} 0 & \zeta_1 \mathrm{d}x^1 + \zeta_2 \mathrm{d}x^2 & \zeta_2 \mathrm{d}x^1 + \zeta_3 \mathrm{d}x^2 \\ \mathrm{d}x^1 & -\kappa_1 \mathrm{d}x^1 - \kappa_2 \mathrm{d}x^2 & -\kappa_2 \mathrm{d}x^1 - \kappa_3 \mathrm{d}x^2 \\ \mathrm{d}x^2 & \kappa_0 \mathrm{d}x^1 + \kappa_1 \mathrm{d}x^2 & \kappa_1 \mathrm{d}x^1 + \kappa_2 \mathrm{d}x^2 \end{pmatrix},$$

with

$$\kappa_0 = \Pi^2_{11}, \quad \kappa_1 = \Pi^2_{12}, \quad \kappa_2 = \Pi^2_{22}, \quad \kappa_3 = -\Pi^1_{22}.$$

Integral manifolds of the EDS

$$\left\langle \theta^0_0, \mathrm{d}\theta^1_0, \mathrm{d}\theta^2_0 \right\rangle, \quad \Theta = \theta^1_0 \wedge \theta^2_0.$$

Conversely integral manifolds locally give coordinate sections.

Weyl metrisability

Lemma. *Suppose* $(M^2, [\nabla])$ *admits a compatible Weyl structure* $[g, \beta]$*, then in a neighbourhood* U_p *of every point* $p \in M$ *there exists a coordinate section* $\sigma_x : U_p \to B$ *which is an integral manifold of*

$$\mathcal{I}' = \left\langle \theta^0_0, \mathrm{d}\theta^1_0, \mathrm{d}\theta^2_0, \theta^1_0 \wedge (3\,\theta^2_1 + \theta^1_2), \theta^2_0 \wedge (\theta^2_1 + 3\,\theta^1_2) \right\rangle, \quad \Theta = \theta^1_0 \wedge \theta^2_0.$$

Conversely every coordinate section $\sigma_x : U \subset M \to B$ *which is an integral manifold of* $(\mathcal{I}', \Omega)$ *gives rise to a Weyl structure* $[g, \beta]$ *on* U *which is compatible with* $[\nabla]$*.*

"Proof". *Locally Weyl metrisability is equivalent to finding coordinates x so that*

$$\kappa_0 = 3\kappa_2, \quad 3\kappa_1 = \kappa_3.$$

Weyl structure is given by $[x^* g_E, -\kappa_3 \mathrm{d}x^1 + \kappa_0 \mathrm{d}x^2]$*.*

The coordinates x are isothermal for $[g]$.

Linear Pfaffian system

Consider $A = B \times \mathbb{R}^4$ with (a_i coordinates on $\mathbb{R}^4$)

$$\begin{aligned}
\varphi^1 &= \theta^1_1 + a_1\theta^1_0 + a_2\theta^2_0, \\
\varphi^2 &= \theta^2_2 - a_1\theta^1_0 - a_2\theta^2_0, \\
\varphi^3 &= \theta^1_2 + a_2\theta^1_0 + 3a_1\theta^2_0, \\
\varphi^4 &= \theta^2_1 - 3a_2\theta^1_0 - a_1\theta^2_0.
\end{aligned}$$

Integral manifolds of the linear Pfaffian system

$$\mathcal{I} = \left\langle \varphi^1, \varphi^2, \varphi^3, \varphi^4 \right\rangle, \Theta = \theta^1_0 \wedge \theta^2_0$$

correspond to the integral manifolds of $(\mathcal{I}', \Theta)$.

$(\mathcal{I}, \Theta)$ is Frobenius, determined and elliptic.

Elliptic PDE theory yields local existence of solutions in the smooth category.

Conformal structure

Think of a conformal structure as a G-structure where G is the Lie group

$$\mathrm{CO}(n) = \left\{a \in \mathrm{GL}(n,\mathbb{R}) \,|\, aa^t = \lambda \mathrm{I}_n,\ \lambda \in \mathbb{R}^+\right\}.$$

For $n = 2$ we have $\mathrm{GL}(1,\mathbb{C}) \simeq \mathrm{CO}^+(2) = \mathrm{CO}(2) \cap \mathrm{GL}^+(2,\mathbb{R})$.

A conformal structure $[g]$ on the oriented surface M is a section of the bundle $\rho : \mathcal{C}(M) \to M$ where

$$\mathcal{C}(M) = \mathcal{F}^+/\mathrm{GL}(1,\mathbb{C})$$

and $\mathcal{F}^+$ is the total space of bundle of positively oriented frames. $\mathcal{C}(M) \simeq \mathcal{J}^+(M)$, the bundle of orientation compatible “twistors” whose fibre at $p \in M$ is

$$\mathcal{J}^+(M)_p = \left\{j \in \mathrm{Aut}(T_pM) \,|\, j^2 = -\mathrm{Id}_{T_pM},\ \omega(v, j(v)) \geq 0 \text{ for } \omega > 0\right\}.$$

Complex structure on $\mathcal{J}(M^2)$

Theorem (Dubois-Violette ’83, O’Brian & Rawnsley ’85)**.** *An affine torsion-free connection ∇ on TM^{2n} induces an almost complex structure $\mathfrak{J}$ on $\mathcal{J}(M)$ which is integrable if and only if the Weyl projective curvature tensor of ∇ vanishes.*

Observation $\mathfrak{J}$ does only depend on $[\nabla]$.

If M is oriented, the subbundle $\kappa : \mathcal{J}^+(M) \to M$ of orientation compatible twistors inherits an almost complex structure as well.

For $n = 1$ we have

$$W_{[\nabla]} = 0,$$

thus for an oriented projective surface $(M, [\nabla])$ the manifold $\mathcal{C}(M)$ is a complex surface.

Weyl metrisability problem

Theorem (M–). *A conformal structure* $[g]$ *on an oriented projective surface* $(M, [\nabla])$ *underlies a* $[\nabla]$*-compatible Weyl structure* $[g, \beta]$ *if and only if* $[g] : M \to \mathcal{C}(M)$ *is a holomorphic curve.*

$[g] : M \to \mathcal{C}(M)$ is a holomorphic curve if

$$(J \circ [g]')(T_pM) = [g]'(T_pM), \quad \forall\, p \in M,$$

where $J : T\mathcal{C}(M) \to T\mathcal{C}(M)$ is the complex structure map of $\mathcal{C}(M)$.

Corollary (M–). *Every projective surface locally admits a compatible Weyl structure.*

Example. $(M, [\nabla]) = (\mathbb{S}^2, [D^{g_0}])$

$$\mathcal{C}(\mathbb{S}^2) \simeq \mathbb{CP}^2 \setminus \mathbb{RP}^2.$$

Base-point projection is given by $\rho_0 : [z] \mapsto [\mathrm{Re}(z) \times \mathrm{Im}(z)]$.

The flat case

Corollary (M–). *The Weyl structures on the 2-sphere whose geodesics are the great circles are in one-to-one correspondence with the smooth quadrics (i.e. a smooth algebraic curves of degree 2) $\mathcal{C} \subset \mathbb{CP}^2$ without real points.*

Related to rectilinear Finsler metrics on S^2 of constant positive Finsler-Gauss curvature as studied by Bryant (1997).

A Weyl structure $[g, \beta]$ on an oriented surface M is said to be positive (or has positive curvature) if

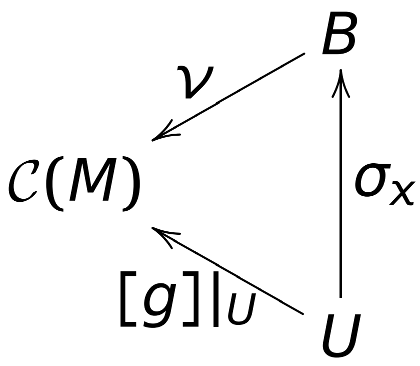
$$(K - \delta\beta)\, dA > 0.$$

Theorem (Bryant, ICM 2006, M–, 2009). *Every oriented positive compact Weyl-Zoll surface gives rise to an oriented $K = 1$ Finsler 2-sphere (unique up to isometry) with 2π-periodic geodesic flow. Conversely every oriented compact $K = 1$ Finsler surface with 2π-periodic geodesic flow gives rise to an oriented positive Weyl-Zoll 2-sphere (unique up to diffeomorphism).*

Sketch of proof

Locally translate $[g] : M \to \mathcal{C}(M)$ being a holomorphic curve into a condition for σ_x for $[g]$-isothermal coordinates x.

Use: σ_x commutes with $[g]$ and $\nu : B \to \mathcal{C}(M)$, $j^2_0\varphi \mapsto [(\varphi_* g_E)_{\varphi(0)}]$ for orientation preserving $[g]$-isothermal coordinates $x : U \to \mathbb{R}^2$.



Lemma. *Let* (X,J) *be a complex surface,* $\mu_1, \mu_2 \in \Omega^{1,0}(X, \mathbb{C})$ *a basis for the* $(1,0)$*-forms and* $f : \Sigma \to X$ *a* 2*-submanifold with*

$$f^*(\mathrm{Re}(\mu_1) \wedge \mathrm{Im}(\mu_1)) \neq 0.$$

Then (f, Σ) *is a holomorphic curve* $\iff f^*(\mu_1 \wedge \mu_2) = 0.$

Relate $(\mathcal{C}(M), J)$ to (B, θ)

Let $\alpha_1 = \theta^1_0 + \mathrm{i}\theta^2_0$ and $\alpha_2 = \left(\theta^1_2 + \theta^2_1\right) + \mathrm{i}\left(\theta^2_2 - \theta^1_1\right)$.

Proposition. *The map $\nu : B \to \mathcal{C}(M)$, $j^2_0\varphi \mapsto [(\varphi_* g_E)_{\varphi(0)}]$ makes B into a principal bundle over $\mathcal{C}(M)$. Moreover $\exists$ a unique complex structure J on $\mathcal{C}(M)$ such that*

$$\mu \in \Omega^{(1,0)}_J(\mathcal{C}(M)) \iff \nu^*\mu = \lambda_1\alpha_1 + \lambda_2\alpha_2, \quad \lambda_i \in C^\infty(B, \mathbb{C}).$$

Example. $(M^2, [\nabla]) = (\mathbb{S}^2, [D^{g_0}])$. *Then* $\nu : \mathrm{SL}(3, \mathbb{R}) \to \mathbb{CP}^2 \setminus \mathbb{RP}^2$

$$(g_1, g_2, g_3) \mapsto [g_1 \times (g_2 + ig_3)]$$

EDS on B, $\mathcal{I} = \left\langle \theta^0_0, \mathrm{d}\theta^1_0, \mathrm{d}\theta^2_0, \mathrm{Re}\,(\alpha_1 \wedge \alpha_2), \mathrm{Im}\,(\alpha_1 \wedge \alpha_2)\right\rangle$

Lemma. *Let $x : U \to \mathbb{R}^2$ be $[g]$-isothermal orientation preserving coordinates. Then $[g]|_U : U \to \mathcal{C}(M)$ is a holomorphic curve if and only if $\sigma_x : U \to B$ satisfies $(\sigma_x)^*\mathcal{I} = 0$.*

EDS for Weyl metrisability

$$\mathcal{I}' = \left\langle \theta^0_0, \mathrm{d}\theta^1_0, \mathrm{d}\theta^2_0, \theta^1_0 \wedge (3\,\theta^2_1 + \theta^1_2), \theta^2_0 \wedge (\theta^2_1 + 3\,\theta^1_2) \right\rangle .$$

EDS for holomorphic curves

$$\mathcal{I} = \left\langle \theta^0_0, \mathrm{d}\theta^1_0, \mathrm{d}\theta^2_0, \mathrm{Re}\,(\alpha_1 \wedge \alpha_2)\,, \mathrm{Im}\,(\alpha_1 \wedge \alpha_2) \right\rangle$$

Finally observe

$$\left(\theta^1_0 \wedge (3\theta^2_1 + \theta^1_2)\right) + \mathrm{i}\left(\theta^2_0 \wedge (\theta^2_1 + 3\theta^1_2)\right) = \alpha_1 \wedge \alpha_2 + 3\mathrm{i}\bar{\alpha}_1 \wedge \theta^0_0 + 2\mathrm{i}\mathrm{d}\bar{\alpha}_1$$

and that

$$D^{g,\beta} \sim D^{g,\beta'} \quad \Rightarrow \quad \beta = \beta'.$$