Rigid Schubert classes in compact Hermitian symmetric spaces

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joint work with Dennis The

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Part A: Question of Borel & Haefliger

1. Compact Hermitian symmetric spaces.
2. Schubert varieties.
3. The motivating question.

CHSS

$X = G/P$ a compact Hermitian symmetric space (CHSS).

Example (Projective space)

$\mathbb{CP}^n = \mathrm{Gr}(1, n+1)$

Example (Grassmannians)

The space $\mathrm{Gr}(k, m)$ of k–dimensional linear subspaces in $\mathbb{C}^m$.

$$
\begin{aligned}
G &= \mathrm{SL}(m, \mathbb{C})\,, \\
P &= \text{stabilizer of fixed } k\text{–plane } \zeta \subset \mathbb{C}^m\,.
\end{aligned}
$$

Irreducible CHSS

Classical

Grassmannian	$\mathrm{Gr}(k, n+1) = \mathrm{SL}_{n+1}/P_k$.
Quadric hypersurfaces	$Q^m = \mathrm{SO}_{m+2}/P_1 \subset \mathbb{P}^{m+1}$.
Lagrangian grassmannians	$\mathrm{LG}(n, 2n) = \mathrm{Sp}_n/P_n$.
Spinor varieties	$X^{n(n-1)/2} = \mathrm{SO}_{2n}/P_n$.

Exceptional

Cayley plane	$X^{16} = E_6/P_6$.
Freudenthal variety	$X^{27} = E_7/P_7$.

CHSS as algebraic varieties

Definition

G a complex semisimple Lie group.

V an irreducible G–representation. That is, $G \subset \mathrm{GL}(V)$.

Fact

There is a unique compact G–orbit $X \subset \mathbb{P}V$.

Definition

P the stabilizer of a point $o \in X$.

Fact

$X \simeq G/P$ is a (smooth) homogeneous variety.

Theorem (E. Cartan)

X admits structure of a CHSS if and only if the isotropy representation of P on T_oX is irreducible.

Schubert varieties

Theorem (B. Kostant 1963)

The classes $\sigma = [S]$ of the Schubert varieties $S \subset X$ form an additive basis of the integral homology $H_\bullet(X)$.

Example (Grassmannian)

Fix $0 < \ell \leq m - k$ and a subspace $W \subset \mathbb{C}^m$ of codimension $k + \ell - 1$.

$$S_{(\ell)} \;=\; \{\zeta \in \mathrm{Gr}(k, m) \mid \dim(E \cap W) \neq 0\}$$

is a Schubert variety of codimension ℓ.

Singular Schubert varieties

Example (Grassmannian)

Fix $0 < \ell \leq m - k$ and a subspace $W \subset \mathbb{C}^m$ of codimension $k + \ell - 1$.

$$S_{(\ell)} \;=\; \{\zeta \in \mathrm{Gr}(k, m) \mid \dim(E \cap W) \neq 0\}$$

is a Schubert variety of codimension ℓ.

Facts

1. If $1 < k < m - 1$, then $S_{(\ell)}$ is **singular**.

Example

If $k = 2$, then $\mathrm{Sing}(S_{(\ell)}) = \mathrm{Gr}(2, W) \subset \mathrm{Gr}(2, n)$.

2. Most Schubert varieties are singular;
3. S is the 'most singular' variety representing $[S]$.

Motivating Question (Borel and Haefliger 1961)

Which Schubert classes σ can be represented by a smooth variety $\mathbf{Y} \subset \mathbf{X}$?

Broader Context

Identification of distinguished representatives of (co)homology classes. Examples include:

1. Hodge Decomposition Theorem: $H^\bullet_{\mathrm{dR}}(M, \mathbb{R}) \simeq \mathcal{H}^\bullet(M, g)$.
2. Calibrated geometry $\rightsquigarrow$ calibrated submanifolds N^k are global minimizers of volume in $[N] \in H_k(M, \mathbb{R})$.
3. Hodge Conjecture (Millennium Prize Problem).

Definition

When $\exists$ smooth Y representing $\sigma = [S]$, we say σ is **smoothable**.

Example

The hypersurface $S_{(1)} \subset \mathrm{Gr}(k, m)$ is a hyperplane section;
Bertini's Theorem $\implies$ $\sigma_{(1)}$ is smoothable.

Theorem (Hartshorne, Rees & Thomas 1974)

1. $\sigma_{(2)} \in H_{14}(\mathrm{Gr}(3, 6))$ can not be represented by any integral linear combination of smooth, oriented submanifolds of (real) codimension four.
2. $\sigma_{(2)} = [Y_1] - [Y_2] \in H_8(\mathrm{Gr}(2, 5))$, but cannot be smoothed.

Part B: What is known

1. Izzet Coskun
2. A differential geometric approach

2.a Maria Walters and Robert Bryant

2.b Jaehyun Hong

Work of Izzet Coskun 2010

Theorem

A (nearly sharp) description of the smoothable Schubert classes in the Grassmannian.

Definition

A Schubert class $[S]$ is **rigid** if the only varieties Y representing the class are the G–translates $g \cdot S$.

Theorem

Identified all rigid Schubert classes in the Grassmannian.

A differential geometric approach

Theorem (Maria Walters$^{\mathrm{Gr}}$ 1997 & Robert Bryant 2001)

The varieties Y with the property that

$$[Y] \;=\; r\sigma\,, \quad \text{for some } r \in \mathbb{Z}\,, \tag{1}$$

are characterized by the Schur differential system (to be defined).

Definition

The Schubert variety S is **Schur rigid** if every irreducible variety Y satisfying (1) is a G–translate $g \cdot S$.

Otherwise, S is **Schur flexible**.

S Schur rigid	$\Longrightarrow$	σ rigid.
S singular and Schur rigid	$\Longrightarrow$	σ **not** smoothable.

BH question can be studied via differential geometry.

Schur flexibility for trivial topological reasons

When $H_{2k}(X)$ is generated by a single Schubert class $\sigma = [S]$,

$$H_{2k}(X) \;=\; \mathbb{Z}\,,$$

every Y^k satisfies $[Y] = r\sigma$. $\quad\Longrightarrow\quad$ S is Schur flexible.

Examples

The following are Schur flexible by topology:

1a. Every Schubert variety of $\mathbb{P}^n$.

1b. Any $S \subsetneq \mathbb{P}^m \subset X$.

2a. Every Schubert variety of Q^{2n-1}.

2b. Any Schubert variety of Q^{2n}, with $\dim S \neq n$.

Remark

There are two Schubert varieties $S \subset Q^{2n}$ of dimension n; they are maximal linear spaces $\mathbb{P}^n$.

Work of Walters and Bryant

Maria Walters, Ph.D. Thesis 1997

Identified first-order obstructions to Schur flexibility for

(A) smooth Schubert varieties in $\mathrm{Gr}(k, m)$, and

(B) codimension two Schubert varieties in $\mathrm{Gr}(2, m)$.

Robert Bryant 2001

Identified first-order obstructions to Schur flexibility for

(1) smooth Schubert varieties in $\mathrm{Gr}(k, m)$ and $\mathrm{LG}(n, 2n)$,

(2) maximal linear subspaces in the classical CHSS,

(3) singular Schubert varieties of low (co)dimension in $\mathrm{Gr}(k, m)$.

Work of Jaehyun Hong

Theorem (Hong 2007)

Let X be an irreducible CHSS, excluding the quadrics of odd dimension. Let $S \subset X$ be a **smooth** Schubert variety, excluding the non-maximal linear subspaces, and $\mathbb{P}^1 \subset \mathrm{LG}(n, 2n)$. Then S is Schur rigid.

Remark

Omissions above are for trivial topological reasons.

Theorem (Hong 2005)

Identified a large class of singular Schubert varieties in $\mathbf{Gr(k, m)}$ for which there exist first-order obstructions to Schur flexibility.

Part C: Main Result – joint with Dennis The

1. Statement and examples.
2. Key tools in Hong's approach.
3. Obstructions to extending Hong's strategy to the general case.
4. Outline of our solution.

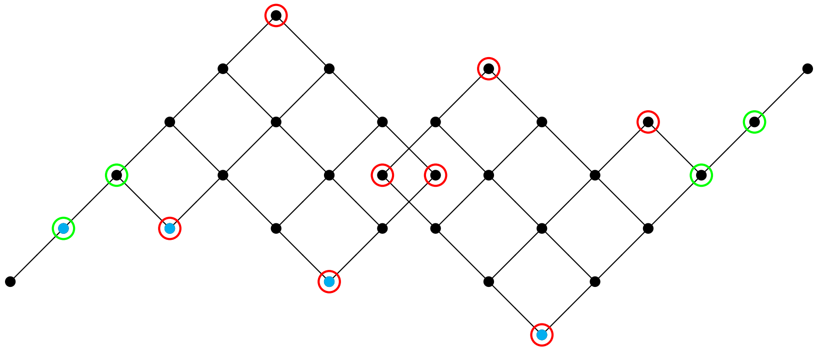
Theorem (Robles - The 2011)

A complete list of the Schubert varieties in a CHSS for which there exist first-order obstructions to Schur flexibility.

Remark

1. List includes all (singular) S for which there exist first-order obstructions to the existence of Y (BH's question).
2. Theorem recovers the results of Walters, Bryant and Hong.
3. A Schubert class appears on this list if and only if its Poincaré dual does.
4. Theorem need not be a complete list of Schur rigid Schubert varieties: there may be higher-order obstructions.
5. The $S_{(\ell)} \subset \mathrm{Gr}(k, m)$ are *not* on the list.

Example: the Lagrangian Grassmannian $X^{15} = \mathrm{LG}(5, 10)$

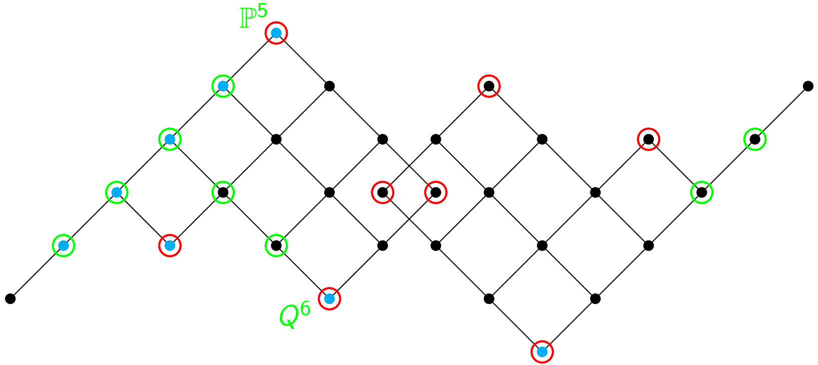


Smooth (proper) Schubert variety.

Schur flexible by topology.

Schur rigid: $\exists$ first-order obstructions to flexibility.

Example: the Spinor variety $X^{15} = D_6/P_6$



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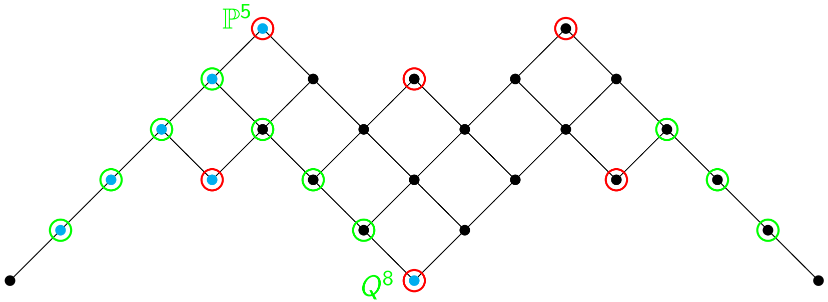

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Schur flexible by topology.

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Hong's approach for smooth S

Two key tools

1. Well-known description of smooth S by connected sub-diagrams of the Dynkin diagram of G.
2. A Lie algebra cohomology $H^1(\sigma)$ arises in analysis.

   First-order obstruction to flexibility is equivalent to the vanishing of a subspace of $H^1(\sigma)$.

   Theorem of Kostant (1961) reduces computation of $H^1(\sigma)$ to Weyl group combinatorics.

Obstructions to generalizing to singular case

1. No analogous description of the singular S.
2. Kostant's Theorem does not apply to $H^1(\sigma)$.

The sine quibus non of our approach

1. Characterization of the Schubert varieties by an integer $0 \leq \mathtt{a}$ and a marking $\mathtt{J}$ of the Dynkin diagram of G.

   This generalizes the descriptions of both
   - the smooth Schubert varieties ($\mathtt{a} = 0$), and
   - the Schubert varieties in $\mathrm{Gr}(k, m)$ by partitions.

2. Construction of an algebraic Laplacian $\Box$ (à la Kostant) with the property that

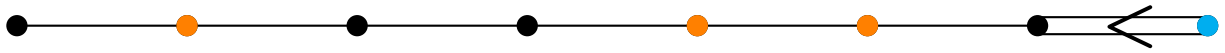
$$\ker \Box \;\simeq\; \text{Lie algebra cohomology.}$$

Characterization of Schubert varieties

Theorem

The Schubert varieties S are characterized by an integer $0 \leq \mathtt{a}(S)$ and a marking $\mathtt{J}(S)$ of the Dynkin diagram of G.

Example ($\mathrm{Sp}_8/P_8 = \mathrm{LG}(8,16)$)



Dynkin diagram of Sp_8. Marking for parabolic P_8.

$\mathtt{J} = \{2,5,6\}$.

$$\mathtt{J} \rightsquigarrow \mathtt{E} \in \mathfrak{p}$$

$$\rightsquigarrow \mathtt{E}\text{–eigenspace decomp.}$$

$$T_oX = \mathfrak{n}_0 \oplus \mathfrak{n}_{-1} \oplus \cdots \oplus \mathfrak{n}_{-\mathtt{A}}$$

$$\mathfrak{n}_0 \oplus \cdots \oplus \mathfrak{n}_{-\mathtt{a}} = T_oS, \quad \text{for some } S.$$

Precise statements for the Lagrangian Grassmannian LG$(n, 2n)$

Theorem (Characterization of Schubert varieties by (a, J))

$\exists$ a bijection between Schubert varieties and pairs (a, J) satisfying

$$\mathrm{J} = \{ \mathrm{j_p} < \cdots < \mathrm{j_1} \} \subset \{1, \ldots, n-1\} \quad \text{and} \quad |\mathrm{J}| = \mathrm{a}, \mathrm{a}+1 .$$

Theorem (First-order obstructions to Schur flexibility)

The S for which $\exists$ first-order obstructions to Schur flexibility are

1. $|\mathrm{J}| = \mathrm{a}$, any J with $1 < \mathrm{j}_\ell - \mathrm{j}_{\ell-1}$ for all $1 \leq \ell \leq \mathrm{p}$;
2. $|\mathrm{J}| = \mathrm{a}+1$, any J with $1 < \mathrm{j}_\ell - \mathrm{j}_{\ell-1}$ for all $2 \leq \ell \leq \mathrm{p}+1$.

For the singular S above $(\mathrm{a} > 0)$, $[S]$ is not smoothable.

Two differential systems on $X = G/P$

Fix $\sigma = [S]$, and $s := \dim S$.

$\mathrm{Gr}(s, TX) =$ Grassmann bundle of tangent s–planes $\subset \mathbb{P}(\bigwedge^s TX)$.

$\mathcal{B}_\sigma =$ sub-bundle of s–planes tangent to $g \cdot S^0$ for some $g \in G$.

$\mathcal{B}_\sigma \subset \mathcal{R}_\sigma := \langle \mathcal{B}_\sigma \rangle \cap \mathrm{Gr}(s, TX)$.

Definition. $Y \subset X$ is an **integral variety** of...

1. the **Schubert system** $\mathcal{B}_\sigma$ if $TY^0 \subset \mathcal{B}_\sigma$.
2. the **Schur system** $\mathcal{R}_\sigma$ if $TY^0 \subset \mathcal{R}_\sigma$.

Schur flexibility for representation theoretic reasons

Recall $\mathcal{B}_\sigma \subset \mathcal{R}_\sigma := \langle \mathcal{B}_\sigma \rangle \cap \mathrm{Gr}(s, TX)$.

$$B_\sigma := \mathcal{B}_{\sigma,o} \qquad R_\sigma := \mathcal{R}_{\sigma,o} .$$

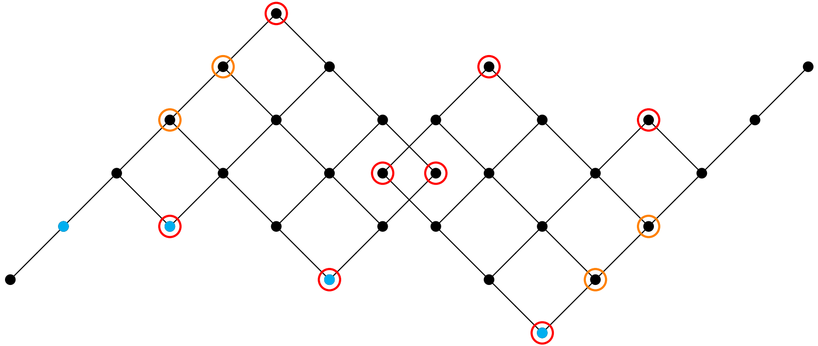
Theorem (Bryant$^{\mathrm{Gr}}$ 2001 & Hong 2007)

If $B_\sigma \subsetneq R_\sigma$, then S is Schur flexible.

Theorem (Robles - The 2011)

A complete list of the Schubert varieties in CHSS with $B_\sigma = R_\sigma$.

Example: the Lagrangian Grassmannian $X^{15} = \mathrm{LG}(5, 10)$



Smooth (proper) Schubert variety.

Schur rigid: $\exists$ first-order obstructions to flexibility.

$B_\sigma = R_\sigma$ (nec. for Schur rigidity), but no first-order obs.

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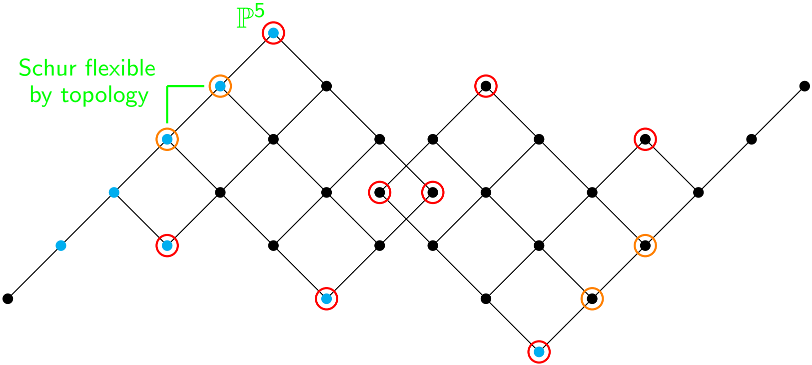

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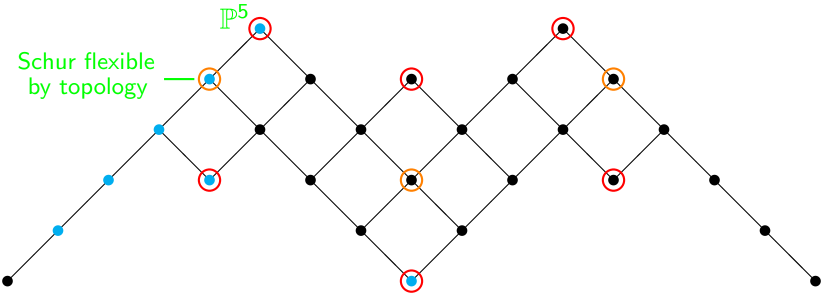

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Proofs: Role of Lie algebra cohomology I

$\exists$ Lie algebra cohomology $H^1(\sigma)$ associated to σ.

Cohomology admits a P–induced graded decomposition

$$H^1(\sigma) \;=\; H^1_0(\sigma) \,\oplus\, H^1_1(\sigma) \,\oplus\, H^1_2(\sigma)\,.$$

To identify the σ for which $B_\sigma \subsetneq R_\sigma$:

1. $B_\sigma \;=\; R_\sigma \quad\Longleftrightarrow\quad T_\tau B_\sigma \;=\; T_\tau R_\sigma$, for $\tau \in B_\sigma$.
2. $T_\tau \mathrm{Gr}(s, T_o X) \;=\; H^1_0(\sigma) \,\oplus\, T_\tau(B_\sigma)$.
3. Compute $H^1_0(\sigma)$ and apply representation theoretic argument.

Proofs: Role of moving frames

$$\begin{array}{ccc} \mathcal{G} & \subset & \mathrm{GL}(V) \\ \downarrow & & \\ \mathcal{R}_\sigma & & \\ \downarrow & & \\ X & \subset & \mathbb{P}V \end{array}$$

Assume $B_\sigma = R_\sigma$. The Schur system $\mathcal{R}_\sigma$ lifts to a linear Pfaffian system (with independence condition) on a frame bundle $\mathcal{G} \simeq G$.

Ingredients: $\mathfrak{s} = \mathfrak{stab}_G(S)\,, \quad \mathfrak{g} = \mathfrak{s} \oplus \mathfrak{s}^\perp;$

P induces $\mathfrak{s}^\perp = \mathfrak{s}^\perp_{-1} \oplus \mathfrak{s}^\perp_0 \oplus \mathfrak{s}^\perp_1\,;$

$\vartheta \in \Omega^1(\mathcal{G}, \mathfrak{g})$ the MC form.

Linear Pfaffian System: $\vartheta_{\mathfrak{s}^\perp_{-1}} = 0.$

Independence Condition: $\det(\vartheta_{\mathfrak{s}}) \neq 0.$

S is Schur rigid if and only if every integral manifold $\mathcal{F} \subset \mathcal{G}$ admits a sub-bundle $\mathcal{F}_0$ on which $\vartheta_{\mathfrak{s}^\perp} = 0$.

Proofs: Role of Lie algebra cohomology II

- Let $\mathcal{F} \subset \mathcal{G}$ be a maximal integral manifold: $\vartheta_{\mathfrak{s}^\perp_{-1}} = 0$.
- S is Schur rigid if and only if $\exists\ \mathcal{F}_0 \subset \mathcal{F}$ on which $\vartheta_{\mathfrak{s}^\perp} = 0$.
- Cohomology: $H^1(\sigma) = H^1_0(\sigma) \oplus H^1_1(\sigma) \oplus H^1_2(\sigma)$.

1. Prolongation $\rightsquigarrow \vartheta_{\mathfrak{s}^\perp_0} = \lambda(\vartheta_{\mathfrak{s}_{-1}})$.

   $H^1_1(\sigma) = 0 \implies \exists$ sub-bundle $\mathcal{F}_1 \subset \mathcal{F}$ on which $\vartheta_{\mathfrak{s}^\perp_0} = 0$.

2. Given $\mathcal{F}_1$, prolongation $\rightsquigarrow \vartheta_{\mathfrak{s}^\perp_1} = \mu(\vartheta_{\mathfrak{s}_{-1}})$.

   $H^1_2(\sigma) = 0 \implies \exists$ sub-bundle $\mathcal{F}_0 \subset \mathcal{F}_1$ on which $\vartheta_{\mathfrak{s}^\perp} = 0$.

Proofs: Computing the Lie algebra cohomology

Step 1: Construction of an algebraic Laplacian $\Box$ (à la Kostant) with the property

$$\ker \Box \ := \ \mathcal{H}^1(\sigma) \ \simeq \ H^1(\sigma)\,.$$

Step 2: Compute $\mathcal{H}^1_0(\sigma)$. (Used to determine when $B_\sigma = R_\sigma$.)

- representation theory, including $(\mathtt{a}, \mathtt{J})$ characterization;
- spectral sequence of filtered complex.

Step 3: Compute $\mathcal{H}^1_+(\sigma)$. (Determines first-order obstructions to flexibility.)

- representation theory, including $(\mathtt{a}, \mathtt{J})$ characterization;
- EDS machinery (torsion & prolongation).

Thank you.

What keeps me awake at night (Open Questions)

1. If $X = \mathrm{Gr}(k, m)$, then $\mathrm{a}(S)$ is the number of irred. components in $\mathrm{Sing}(S)$.
   If $X = \mathrm{LG}(n, 2n)$, then $\lceil \frac{1}{2}\mathrm{a}(S) \rceil$ is the number of irred. components in $\mathrm{Sing}(S)$.

   What is the relationship between $\mathrm{a}(S)$ and $\mathrm{Sing}(S)$ in general?

2. Can the $(\mathrm{a}, \mathtt{J})$ characterization be used to extend Coskun's results to arbitrary CHSS?

3. Do there exist higher-order obstructions to flexibility?

4. Characterize the Y satisfying $[Y] = r[S]$.