Lie Pseudo–Groups

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Sur la théorie, si importante sans doute, mais pour nous si obscure, des ≪groupes de Lie infinis≫, nous ne savons rien que ce qui trouve dans les mémoires de Cartan, première exploration à travers une jungle presque impénétrable; mais celle-ci menace de se refermer sur les sentiers déjà tracés, si l'on ne procède bientôt à un indispensable travail de défrichement.

— André Weil, 1947

What's the Difficulty with Infinite–Dimensional Groups?

- Lie invented Lie groups to study symmetry and solution of differential equations.

♢ In Lie's time, there were no abstract Lie groups. All groups were realized by their action on a space.

♠ Therefore, Lie saw no essential distinction between finite-dimensional and infinite-dimensional group actions.

However, with the advent of abstract Lie groups, the two subjects have gone in radically different directions.

♡ The general theory of finite-dimensional Lie groups has been rigorously formalized and applied.

♣ But there is still no generally accepted abstract object that represents an infinite-dimensional Lie pseudo-group!

Ehresmann's Trinity

1953:

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- Lie pseudo-groups

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- Jets

Ehresmann's Trinity

1953:

- Lie pseudo-groups
- Jets
- Groupoids

Lie Pseudo-groups — History

- Lie, Medolaghi, Vessiot
- É. Cartan
- Ehresmann, Libermann
- Kuranishi, Spencer, Singer, Sternberg, Guillemin, Kumpera, . . .

Lie Pseudo-groups in Applications

- Relativity
- Noether's (Second) Theorem
- Gauge theory and field theories:
  Maxwell, Yang–Mills, conformal, string, ...
- Fluid mechanics, metereology:
  Navier–Stokes, Euler, boundary layer, quasi-geostropic, ...
- Linear and linearizable PDEs
- Solitons (in $2+1$ dimensions): K–P, Davey-Stewartson, ...
- Kac–Moody
- Morphology and shape recognition
- Control theory
- Geometric numerical integration
- Cartan equivalence problems
- *Lie groups*!

Lie Pseudo-groups — Moving Frames

♢ Motivation: To develop an algorithmic invariant calculus for Lie group and pseudo-group actions. Classify and construct differential invariants — including their generators and syzygies — invariant differential forms, invariant differential operators, invariant differential equations, invariant variational problems, etc.

♠ Tools: The equivariant approach to moving frames — which can be implemented for arbitrary Lie group and most Lie pseudo-group actions — along with the induced invariant variational bicomplex.

♡ Additional benefits: A new, elementary approach to the structure theory for Lie pseudo-groups, including explicit construction of Maurer–Cartan forms and direct, elementary determination of structure equations from the infinitesimal generators.

$\Longrightarrow$ PJO, Pohjanpelto, Cheh, Itskov, Valiquette

Pseudo-groups

M — analytic (smooth) manifold

Definition. A pseudo-group is a collection of local analytic diffeomorphisms

$$\phi \colon \operatorname{dom}\phi \subset M \to M \text{ such that}$$

- *Identity*: $\mathbf{1}_M \in \mathcal{G}$
- *Inverses*: $\phi^{-1} \in \mathcal{G}$
- *Restriction*: $U \subset \operatorname{dom}\phi \implies \phi \,|\, U \in \mathcal{G}$
- *Continuation*: $\operatorname{dom}\phi = \bigcup U_\kappa$ and $\phi \,|\, U_\kappa \in \mathcal{G} \implies \phi \in \mathcal{G}$
- *Composition*: $\operatorname{im}\phi \subset \operatorname{dom}\psi \implies \psi \circ \phi \in \mathcal{G}$

The Diffeomorphism Pseudo-group

M — m-dimensional manifold

$\mathcal{D} = \mathcal{D}(M)$ — pseudo-group of all local analytic diffeomorphisms

$$Z = \phi(z)$$

$$\begin{cases} z = (z^1, \ldots, z^m) & \text{— source coordinates} \\ Z = (Z^1, \ldots, Z^m) & \text{— target coordinates} \end{cases}$$

$$\begin{cases} L_\psi(\phi) = \psi \circ \phi & \text{— left action} \\ R_\psi(\phi) = \phi \circ \psi^{-1} & \text{— right action} \end{cases}$$

Jets

For $0 \leq n \leq \infty$:

Given a smooth map $\phi\colon M \longrightarrow M$, written in local coordinates as $Z = \phi(z)$, let $\mathrm{j}_n\phi|_z$ denote its n–jet at $z \in M$, i.e., its n^{th} order Taylor polynomial or series based at z.

$\mathrm{J}^n(M, M)$ is the n^{th} order jet bundle, whose points are the jets.

Local coordinates on $\mathrm{J}^n(M, M)$:

$$(z, Z^{(n)}) = (\ \ldots\ z^a\ \ldots\ Z^b\ \ldots\ Z^b_A\ \ldots\ ), \qquad Z^b_A = \frac{\partial^k Z^b}{\partial z^{a_1} \cdots \partial z^{a_k}}$$

Diffeomorphism Jets

The $n^{\rm th}$ order diffeomorphism jet bundle is the subbundle

$$\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M) \subset \mathrm{J}^n(M, M)$$

consisting of $n^{\rm th}$ order jets of local diffeomorphisms $\phi\colon M \to M$.

The Inverse Function Theorem tells us that $\mathcal{D}^{(n)}$ is defined by the non-vanishing of the Jacobian determinant:

$$\det(\, Z^a_b \,) = \det(\, \partial Z^a / \partial z^b \,) \neq 0$$

★ $\mathcal{D}^{(n)}$ forms a groupoid under
composition of Taylor polynomials/series.

Groupoid Structure

Double fibration:

$$\begin{array}{ccc} & \mathcal{D}^{(n)} & \\ {\scriptstyle \boldsymbol{\sigma}^{(n)}} \swarrow & & \searrow {\scriptstyle \boldsymbol{\tau}^{(n)}} \\ M & & M \end{array}$$

$$\boldsymbol{\sigma}^{(n)}(z, Z^{(n)}) = z \qquad \text{— source map}$$

$$\boldsymbol{\tau}^{(n)}(z, Z^{(n)}) = Z \qquad \text{— target map}$$

You are only allowed to multiply $h^{(n)} \cdot g^{(n)}$ if

$$\boldsymbol{\sigma}^{(n)}(h^{(n)}) = \boldsymbol{\tau}^{(n)}(g^{(n)})$$

◇ Composition of Taylor polynomials/series is well-defined only when the source of the second matches the target of the first.

One-dimensional case: $M = \mathbb{R}$

Source coordinate: x Target coordinate: X

Local coordinates on $\mathcal{D}^{(n)}(\mathbb{R})$

$$g^{(n)} = (x, X, X_x, X_{xx}, X_{xxx}, \ \ldots \ , X_n)$$

Diffeomorphism jet:

$$X[\![h]\!] = X + X_x\, h + \tfrac{1}{2}\, X_{xx}\, h^2 + \tfrac{1}{6}\, X_{xxx}\, h^3 + \ \cdots$$

$\Longrightarrow$ Taylor polynomial/series at a source point x

Groupoid multiplication of diffeomorphism jets:

$$(X, \mathbf{X}, \mathbf{X}_X, \mathbf{X}_{XX}, \ \ldots\ ) \cdot (x, X, X_x, X_{xx}, \ \ldots\ )$$
$$= (x, \mathbf{X}, \mathbf{X}_X\, X_x, \mathbf{X}_X\, X_{xx} + \mathbf{X}_{XX}\, X_x^2, \ \ldots\ )$$

$\Longrightarrow$ Composition of Taylor polynomials/series

- The groupoid multiplication (or Taylor composition) is only defined when the source coordinate X of the first multiplicand matches the target coordinate X of the second.
- The higher order terms are expressed in terms of Bell polynomials according to the general Fàa–di–Bruno formula.

Pseudo-group Jets

Any pseudo-group $\mathcal{G} \subset \mathcal{D}$ defines

a Lie sub-groupoid $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$.

Definition. $\mathcal{G}$ is regular if, for $n \gg 0$, its jets $\boldsymbol{\sigma} : \mathcal{G}^{(n)} \longrightarrow M$ form an embedded subbundle of $\boldsymbol{\sigma} : \mathcal{D}^{(n)} \longrightarrow M$ and the projection $\pi_n^{n+1} : \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a fibration.

Definition. A regular, analytic pseudo-group $\mathcal{G}$ is called a Lie pseudo-group of order $n^\star \geq 1$ if *every* local diffeomorphism $\phi \in \mathcal{D}$ satisfying $\mathrm{j}_{n^\star}\phi \subset \mathcal{G}^{(n^\star)}$ belongs it: $\phi \in \mathcal{G}$.

In local coordinates, $\mathcal{G}^{(n^\star)} \subset \mathcal{D}^{(n^\star)}$ forms a system of differential equations

$$F^{(n^\star)}(z, Z^{(n^\star)}) = 0$$

called the determining system of the pseudo-group. The Lie condition requires that *every* local solution to the determining system belongs to the pseudo-group.

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What about integrability/involutivity?

Lemma. In the analytic category, for sufficiently large $n \gg 0$ the determining system $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ of a regular pseudo-group is an involutive system of partial differential equations.

Proof: regularity + Cartan–Kuranishi + local solvability.

Lie Completion of a Pseudo-group

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Theorem. $\mathcal{G}$ and $\overline{\mathcal{G}}$ have the same differential invariants, the same invariant differential forms, etc.

★ Thus, for local geometry, there is no loss in generality assuming all (regular) pseudo-groups are Lie pseudo-groups!

A Non-Lie Pseudo-group

$$X = \phi(x) \qquad Y = \phi(y) \qquad \text{where} \qquad \phi \in \mathcal{D}(\mathbb{R})$$

On the off-diagonal set $M = \{\, (x, y) \,|\, x \neq y \,\}$, the pseudo-group $\mathcal{G}$ is regular of order 1, and $\mathcal{G}^{(1)} \subset \mathcal{D}^{(1)}$ is defined by the first order determining system

$$X_y = Y_x = 0 \qquad X_x, Y_y \neq 0$$

The general solution to the determining system $\mathcal{G}^{(1)}$ forms the Lie completion $\overline{\mathcal{G}}$:

$$X = \phi(x) \qquad Y = \psi(y) \qquad \text{where} \qquad \phi, \psi \in \mathcal{D}(\mathbb{R})$$

Structure of Lie Pseudo-groups

Recall:

The structure of a finite-dimensional Lie group G is specified by its Maurer–Cartan forms — a basis $\mu^1, \ldots, \mu^r$ for the right-invariant one-forms:

$$d\mu^k = \sum_{i<j} C^k_{ij}\, \mu^i \wedge \mu^j$$

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Cartan: Use exterior differential systems and prolongation to determine the structure equations.

We propose a direct approach based on the following observation:

The Maurer–Cartan forms for a Lie group and hence Lie pseudo-group can be identified with the right-invariant one-forms on the jet groupoid $\mathcal{G}^{(\infty)}$.

The structure equations can be determined immediately from the infinitesimal determining equations.

The Variational Bicomplex

The differential one-forms on an infinite jet bundle split into two types:

- horizontal forms
- contact forms

Consequently, the exterior derivative on $\mathcal{D}^{(\infty)}$ splits

$$d = d_M + d_G$$

into horizontal (manifold) and contact (group) components, leading to the variational bicomplex structure on the algebra of differential forms on $\mathcal{D}^{(\infty)}$.

For the diffeomorphism jet bundle

$$\mathcal{D}^{(\infty)} \subset \mathrm{J}^{\infty}(M, M)$$

Local coordinates:

$$\underbrace{z^1, \ \ldots \ z^m,}_{\text{source}} \quad \underbrace{Z^1, \ \ldots \ Z^m,}_{\text{target}} \quad \underbrace{\ldots \ Z^b_A, \ \ldots}_{\text{jet}}$$

Horizontal forms:

$$dz^1, \ \ldots \ , dz^m$$

Basis contact forms:

$$\Theta^b_A = d_G \, Z^b_A = dZ^b_A - \sum_{a=1}^{m} Z^a_{A,a} \, dz^a$$

One-dimensional case: $M = \mathbb{R}$

Local coordinates on $\mathcal{D}^{(\infty)}(\mathbb{R})$

$$(x, X, X_x, X_{xx}, X_{xxx}, \ \ldots \ , X_n, \ \ldots \ )$$

Horizontal form:

$$dx$$

Contact forms:

$$\Theta = dX - X_x \, dx$$

$$\Theta_x = dX_x - X_{xx} \, dx$$

$$\Theta_{xx} = dX_{xx} - X_{xxx} \, dx$$

$$\vdots$$

Maurer–Cartan Forms

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Key observation: Since the right action only affects source coordinates, the target coordinate functions Z^a are right-invariant.

Thus, when we decompose

$$dZ^a = \underbrace{\sigma^a}_{\text{horizontal}} + \underbrace{\mu^a}_{\text{contact}}$$

both components σ^a, μ^a are right-invariant one forms.

Invariant horizontal forms:

$$\sigma^a = d_M Z^a = \sum_{b=1}^{m} Z^a_b \, dz^b$$

Dual invariant total differentiation operators:

$$\mathbb{D}_{Z^a} = \sum_{b=1}^{m} (\, Z^a_b \,)^{-1} \, \mathbb{D}_{z^b}$$

Thus, the invariant contact forms μ^b_A are obtained by invariant differentiation of the order zero contact forms:

$$\mu^b = d_G Z^b = \Theta^b = dZ^b - \sum_{a=1}^{m} Z^b_a \, dz^a$$

$$\mu^b_A = \mathbb{D}^A_Z \mu^b = \mathbb{D}_{Z^{a_1}} \cdots \mathbb{D}_{Z^{a_n}} \mu^b \qquad b = 1, \ldots, m, \ \#A \geq 0$$

One-dimensional case: $M = \mathbb{R}$

Contact forms:

$$\Theta = dX - X_x \, dx$$

$$\Theta_x = \mathbb{D}_x \Theta = dX_x - X_{xx} \, dx$$

$$\Theta_{xx} = \mathbb{D}_x^2 \Theta = dX_{xx} - X_{xxx} \, dx$$

Right-invariant horizontal form:

$$\sigma = d_M \, X = X_x \, dx$$

Invariant differentiation:

$$\mathbb{D}_X = \frac{1}{X_x} \, \mathbb{D}_x$$

Invariant contact forms:

$$\mu = \Theta = dX - X_x \, dx$$

$$\mu_X = \mathbb{D}_X \mu = \frac{\Theta_x}{X_x} = \frac{dX_x - X_{xx} \, dx}{X_x}$$

$$\mu_{XX} = \mathbb{D}_X^2 \mu = \frac{X_x \, \Theta_{xx} - X_{xx} \, \Theta_x}{X_x^3}$$

$$= \frac{X_x \, dX_{xx} - X_{xx} \, dX_x + (X_{xx}^2 - X_x X_{xxx}) \, dx}{X_x^3}$$

$$\vdots$$

$$\mu_n = \mathbb{D}_X^n \mu$$

The Structure Equations for the Diffeomorphism Pseudo–group

$$d\mu^b_A = \sum C^{b,B,C}_{A,c,d}\ \mu^c_B \wedge \mu^d_C$$

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$$d\mu_A^b = \sum C_{A,c,d}^{b,B,C}\, \mu_B^c \wedge \mu_C^d$$

Formal Maurer–Cartan series:

$$\mu^b[\![\, H \,]\!] = \sum_A \frac{1}{A!}\, \mu_A^b\, H^A$$

$$H = (H^1, \ldots, H^m) \ \text{— formal parameters}$$

$$d\mu[\![\, H \,]\!] = \nabla\mu[\![\, H \,]\!] \wedge (\, \mu[\![\, H \,]\!] - dZ \,)$$

$$d\sigma = -\, d\mu[\![\, 0 \,]\!] = \nabla\mu[\![\, 0 \,]\!] \wedge \sigma$$

One-dimensional case: $M = \mathbb{R}$

Structure equations:

$$d\sigma = \mu_X \wedge \sigma \qquad d\mu[\![\, H \,]\!] = \frac{d\mu}{dH}[\![\, H \,]\!] \wedge (\mu[\![\, H \,]\!] - dZ)$$

where

$$\sigma = X_x \, dx = dX - \mu$$

$$\mu[\![\, H \,]\!] = \mu + \mu_X H + \tfrac{1}{2}\mu_{XX} H^2 + \cdots$$

$$\mu[\![\, H \,]\!] - dZ = -\sigma + \mu_X H + \tfrac{1}{2}\mu_{XX} H^2 + \cdots$$

$$\frac{d\mu}{dH}[\![\, H \,]\!] = \mu_X + \mu_{XX} H + \tfrac{1}{2}\mu_{XXX} H^2 + \cdots$$

In components:

$$d\sigma = \mu_1 \wedge \sigma$$

$$d\mu_n = -\,\mu_{n+1} \wedge \sigma \;+\; \sum_{i=0}^{n-1} \binom{n}{i}\, \mu_{i+1} \wedge \mu_{n-i}$$

$$= \sigma \wedge \mu_{n+1} - \sum_{j=1}^{\left[\frac{n+1}{2}\right]} \frac{n-2j+1}{n+1} \binom{n+1}{j}\, \mu_j \wedge \mu_{n+1-j}.$$

$\Longrightarrow$ Cartan

The Maurer–Cartan Forms for a Lie Pseudo-group

The Maurer–Cartan forms for a pseudo-group $\mathcal{G} \subset \mathcal{D}$ are obtained by restricting the diffeomorphism Maurer–Cartan forms σ^a, μ^b_A to $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$.

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⋆⋆ The resulting one-forms are no longer linearly independent, but the dependencies can be determined directly from the infinitesimal generators of $\mathcal{G}$.

Infinitesimal Generators

$\mathfrak{g}$ — Lie algebra of infinitesimal generators of the pseudo-group $\mathcal{G}$

$z = (x, u)$ — local coordinates on M

Vector field:

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha \frac{\partial}{\partial u^\alpha}$$

Vector field jet:

$$\mathrm{j}_n \mathbf{v} \longmapsto \zeta^{(n)} = (\ \ldots \ \zeta^b_A \ \ldots \ )$$

$$\zeta^b_A = \frac{\partial^{\# A} \zeta^b}{\partial z^A} = \frac{\partial^k \zeta^b}{\partial z^{a_1} \cdots \partial z^{a_k}}$$

The infinitesimal generators of $\mathcal{G}$ are the solutions to the

infinitesimal determining equations

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- If $\mathcal{G}$ is the symmetry group of a system of differential equations, then $(*)$ is the (involutive completion of) the usual Lie determining equations for the symmetry group.

Theorem. The Maurer–Cartan forms on $\mathcal{G}^{(\infty)}$ satisfy the invariantized infinitesimal determining equations

$$\mathcal{L}(\ \ldots\ Z^a\ \ldots\ \mu_A^b\ \ldots\ ) = 0 \qquad (\star\star)$$

obtained from the infinitesimal determining equations

$$\mathcal{L}(\ \ldots\ z^a\ \ldots\ \zeta_A^b\ \ldots\ ) = 0 \qquad (\star)$$

by replacing

- source variables z^a by target variables Z^a
- derivatives of vector field coefficients ζ_A^b by right-invariant Maurer–Cartan forms μ_A^b

The Structure Equations for a Lie Pseudo-group

Theorem. The structure equations for the pseudo-group $\mathcal{G}$ are obtained by restricting the universal diffeomorphism structure equations

$$d\mu[\![\,H\,]\!] = \nabla\mu[\![\,H\,]\!] \wedge (\,\mu[\![\,H\,]\!] - dZ\,)$$

to the solution space of the linear algebraic system

$$\mathcal{L}(\;\ldots\;Z^a\;\ldots\;\mu^b_A\;\ldots\;) = 0.$$

$\Longrightarrow$ *symmetry groups of differential equations*

Comparison of Structure Equations

If the action is transitive, then our structure equations are isomorphic to Cartan's. However, this is not true for intransitive pseudo-groups. Whose structure equations are "correct"?

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- For finite-dimensional intransitive Lie group actions, Cartan's pseudo-group structure equations do not coincide with the standard Maurer–Cartan equations. Ours do (upon restriction to a source fiber).

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- For finite-dimensional intransitive Lie group actions, Cartan's pseudo-group structure equations do not coincide with the standard Maurer–Cartan equations. Ours do (upon restriction to a source fiber).
- Cartan's structure equations for isomorphic pseudo-groups can be non-isomorphic. Ours are always isomorphic.

Lie–Kumpera Example

$$\boxed{X = f(x) \qquad\qquad U = \frac{u}{f'(x)}}$$

Infinitesimal generators:

$$\mathbf{v} = \xi \, \frac{\partial}{\partial x} + \varphi \, \frac{\partial}{\partial u} = \xi(x) \, \frac{\partial}{\partial x} - \xi'(x) \, u \, \frac{\partial}{\partial u}$$

Linearized determining system

$$\xi_x = -\,\frac{\varphi}{u} \qquad\qquad \xi_u = 0 \qquad\qquad \varphi_u = \frac{\varphi}{u}$$

Maurer–Cartan forms:

$$\sigma = \frac{u}{U}\, dx = f_x\, dx, \qquad \tau = U_x\, dx + \frac{U}{u}\, du = \frac{-u\, f_{xx}\, dx + f_x\, du}{f_x{}^2}$$

$$\mu = dX - \frac{U}{u}\, dx = df - f_x\, dx, \qquad \nu = dU - U_x\, dx - \frac{U}{u}\, du = -\frac{u}{f_x{}^2}\,(\, df_x - f_{xx}\, dx\,)$$

$$\mu_X = \frac{du}{u} - \frac{dU - U_x\, dx}{U} = \frac{df_x - f_{xx}\, dx}{f_x}, \qquad \mu_U = 0$$

$$\nu_X = \frac{U}{u}\,(dU_x - U_{xx}\, dx) - \frac{U_x}{u}\,(dU - U_x\, dx)$$

$$= -\frac{u}{f_x{}^3}(\, df_{xx} - f_{xxx}\, dx\,) + \frac{u\, f_{xx}}{f_x{}^4}(\, df_x - f_{xx}\, dx\,)$$

$$\nu_U = -\frac{du}{u} + \frac{dU - U_x\, dx}{U} = -\frac{df_x - f_{xx}\, dx}{f_x}$$

First order linearized determining system:

$$\xi_x = -\frac{\varphi}{u} \qquad \xi_u = 0 \qquad \varphi_u = \frac{\varphi}{u}$$

First order Maurer–Cartan determining system:

$$\mu_X = -\frac{\nu}{U} \qquad \mu_U = 0 \qquad \nu_U = \frac{\nu}{U}$$

Substituting into the full diffeomorphism structure equations yields the (first order) structure equations:

$$d\mu = -\,d\sigma = \frac{\nu \wedge \sigma}{U}, \qquad d\nu = -\,\nu_X \wedge \sigma \; - \; \frac{\nu \wedge \tau}{U}$$

$$d\nu_X = -\,\nu_{XX} \wedge \sigma \; - \; \frac{\nu_X \wedge (\tau + 2\,\nu)}{U}$$

Essential Invariants

- The pseudo-group structure equations live on the bundle $\boldsymbol{\tau} : \mathcal{G}^{(\infty)} \longrightarrow M$, and the structure coefficients C^i_{jk} constructed above may vary from point to point.

♡ In the case of a finite-dimensional Lie group action, $\mathcal{G}^{(\infty)} \simeq G \times M$, and this means the basis of Maurer–Cartan forms on each fiber of $\mathcal{G}^{(\infty)}$ is varying with the target point $Z \in M$. However, we can always make a Z–dependent change of basis to make the structure coefficients constant.

⋆ However, for infinite-dimensional pseudo-groups, it may not be possible to find such a change of Maurer–Cartan basis, leading to the concept of essential invariants.

Action of Pseudo-groups on Submanifolds a.k.a. Solutions of Differential Equations

$\mathcal{G}$ — Lie pseudo-group acting on p-dimensional submanifolds:

$$N = \{\, u = f(x) \,\} \subset M$$

For example, $\mathcal{G}$ may be the symmetry group of a system of differential equations

$$\Delta(x, u^{(n)}) = 0$$

and the submanifolds are the graphs of solutions $u = f(x)$.

Goal: Understand $\mathcal{G}$–invariant objects (moduli spaces)

Prolongation

$\mathrm{J}^n = \mathrm{J}^n(M,p)$ — n^{th} order submanifold jet bundle

Local coordinates :

$$z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )$$

Prolonged action of $\mathcal{G}^{(n)}$ on submanifolds:

$$(x, u^{(n)}) \quad \longmapsto \quad (X, \widehat{U}^{(n)})$$

Coordinate formulae:

$$\widehat{U}^\alpha_J = F^\alpha_J(x, u^{(n)}, g^{(n)})$$

$\Longrightarrow$ Implicit differentiation.

Moving Frames

In the finite-dimensional Lie group case, a moving frame is defined as an equivariant map

$$\rho^{(n)} : \mathrm{J}^n \longrightarrow G$$

However, we do not have an appropriate abstract object to represent our pseudo-group $\mathcal{G}$.

Consequently, the moving frame will be an equivariant section

$$\rho^{(n)} : \mathrm{J}^n \longrightarrow \mathcal{H}^{(n)}$$

of the pulled-back pseudo-group jet groupoid:

$$\begin{array}{ccc} \mathcal{G}^{(n)} & & \mathcal{H}^{(n)} \\ \downarrow & & \downarrow \\ M & \longleftarrow & \mathrm{J}^n . \end{array}$$

Moving Frames for Pseudo–Groups

Definition. A (right) *moving frame* of *order* n is a right-equivariant section $\rho^{(n)} : V^n \to \mathcal{H}^{(n)}$ defined on an open subset $V^n \subset \mathrm{J}^n$.

$\Longrightarrow$ Groupoid action.

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$\Longrightarrow$ Groupoid action.

Proposition. A moving frame of order n exists if and only if $\mathcal{G}^{(n)}$ acts *freely* and regularly.

Freeness

For Lie group actions, freeness means trivial isotropy:

$$G_z = \{ g \in G \mid g \cdot z = z \} = \{e\}.$$

For infinite-dimensional pseudo-groups, this definition cannot work, and one must restrict to the transformation jets of order n, using the n^{th} order isotropy subgroup:

$$\mathcal{G}^{(n)}_{z^{(n)}} = \left\{ g^{(n)} \in \mathcal{G}^{(n)}_z \;\middle|\; g^{(n)} \cdot z^{(n)} = z^{(n)} \right\}$$

Definition. At a jet $z^{(n)} \in \mathrm{J}^n$, the pseudo-group $\mathcal{G}$ acts

- freely if $\mathcal{G}^{(n)}_{z^{(n)}} = \{ \mathbf{1}^{(n)}_z \}$
- locally freely if
  - $\mathcal{G}^{(n)}_{z^{(n)}}$ is a discrete subgroup of $\mathcal{G}^{(n)}_z$
  - the orbits have dimension $r_n = \dim \mathcal{G}^{(n)}_z$

$\Longrightarrow$ Kumpera's growth bounds on Spencer cohomology.

Persistence of Freeness

Theorem. If $n \geq 1$ and $\mathcal{G}^{(n)}$ acts (locally) freely at $z^{(n)} \in \mathrm{J}^n$, then it acts (locally) freely at any $z^{(k)} \in \mathrm{J}^k$ with $\widetilde{\pi}^k_n(z^{(k)}) = z^{(n)}$ for all $k > n$.

The Normalization Algorithm

To construct a moving frame :

I. Compute the prolonged pseudo-group action

$$u_K^\alpha \quad \longmapsto \quad U_K^\alpha = F_K^\alpha(x, u^{(n)}, g^{(n)})$$

by implicit differentiation.

II. Choose a cross-section to the pseudo-group orbits:

$$u_{J_\kappa}^{\alpha_\kappa} = c_\kappa, \qquad \kappa = 1, \ldots, r_n = \text{fiber dim } \mathcal{G}^{(n)}$$

III. Solve the normalization equations

$$U_{J_\kappa}^{\alpha_\kappa} = F_{J_\kappa}^{\alpha_\kappa}(x, u^{(n)}, g^{(n)}) = c_\kappa$$

for the n^{th} order pseudo-group parameters

$$g^{(n)} = \rho^{(n)}(x, u^{(n)})$$

IV. Substitute the moving frame formulas into the unnormalized jet coordinates $u_K^\alpha = F_K^\alpha(x, u^{(n)}, g^{(n)})$. The resulting functions form a complete system of n^{th} order differential invariants

$$I_K^\alpha(x, u^{(n)}) = F_K^\alpha(x, u^{(n)}, \rho^{(n)}(x, u^{(n)}))$$

Lie–Tresse–Kumpera Example

$$\boxed{X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}}$$

Horizontal coframe

$$d_H X = f_x \, dx, \qquad d_H Y = dy,$$

Implicit differentiations

$$\mathrm{D}_X = \frac{1}{f_x} \mathrm{D}_x, \qquad \mathrm{D}_Y = \mathrm{D}_y.$$

Prolonged pseudo-group transformations on surfaces $S \subset \mathbb{R}^3$:

$$X = f \qquad Y = y \qquad U = \frac{u}{f_x}$$

$$U_X = \frac{u_x}{f_x^{\,2}} - \frac{u\, f_{xx}}{f_x^{\,3}} \qquad U_Y = \frac{u_y}{f_x}$$

$$U_{XX} = \frac{u_{xx}}{f_x^{\,3}} - \frac{3u_x\, f_{xx}}{f_x^{\,4}} - \frac{u\, f_{xxx}}{f_x^{\,4}} + \frac{3u\, f_{xx}^{\,2}}{f_x^{\,5}}$$

$$U_{XY} = \frac{u_{xy}}{f_x^{\,2}} - \frac{u_y\, f_{xx}}{f_x^{\,3}} \qquad U_{YY} = \frac{u_{yy}}{f_x}$$

$f, f_x, f_{xx}, f_{xxx}, \ldots$ — pseudo-group parameters

$\Longrightarrow$ action is free at every order.

Coordinate cross-section

$$X = f = 0, \quad U = \frac{u}{f_x} = 1, \quad U_X = \frac{u_x}{f_x^{\,2}} - \frac{u\, f_{xx}}{f_x^{\,3}} = 0, \quad U_{XX} = \cdots = 0.$$

Moving frame

$$f = 0, \qquad f_x = u, \qquad f_{xx} = u_x, \qquad f_{xxx} = u_{xx}.$$

Differential invariants

$$U_Y = \frac{u_y}{f_x} \quad \longmapsto \quad J = \iota(u_y) = \frac{u_y}{u}$$

$$U_{XY} = \cdots \quad \longmapsto \quad J_1 = \iota(u_{xy}) = \frac{u u_{xy} - u_x u_y}{u^3}$$

$$U_{YY} = \cdots \quad \longmapsto \quad J_2 = \iota(u_{xy}) = \frac{u_{yy}}{u}$$

$$U_{XXY} \longmapsto J_3 = \iota(u_{xxy}) \qquad U_{XYY} \longmapsto J_4 = \iota(u_{xyy}) \qquad U_{YYY} \longmapsto J_5 = \iota(u_{yyy})$$

Invariant horizontal forms

$$d_H X = f_x \, dx \longmapsto u \, dx, \qquad d_H Y = dy \longmapsto dy,$$

Invariant differentiations

$$\mathcal{D}_1 = \frac{1}{u} \, \mathrm{D}_x \qquad \mathcal{D}_2 = \mathrm{D}_y$$

Higher order differential invariants: $\mathcal{D}_1^m \, \mathcal{D}_2^n \, J$

$$J_{,1} = \mathcal{D}_1 J = \frac{u u_{xy} - u_x u_y}{u^3} = J_1,$$

$$J_{,2} = \mathcal{D}_2 J = \frac{u u_{yy} - u_y^2}{u^2} = J_2 - J^2.$$

Recurrence formulae:

$$\mathcal{D}_1 J = J_1, \qquad \mathcal{D}_2 J = J_2 - J^2,$$

$$\mathcal{D}_1 J_1 = J_3, \qquad \mathcal{D}_2 J_1 = J_4 - 3 \, J \, J_1,$$

$$\mathcal{D}_1 J_2 = J_4, \qquad \mathcal{D}_2 J_2 = J_5 - J \, J_2,$$