The pentagram map and generalizations: integrable discretizations of AGD flows

Gloria Marí Beffa

Montreal, June, 2011

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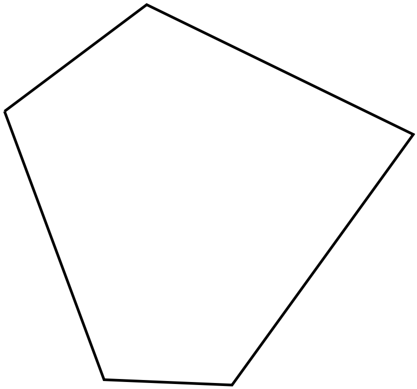

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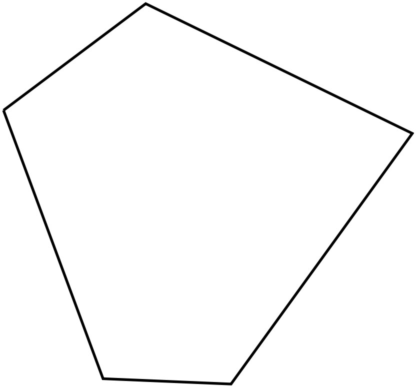

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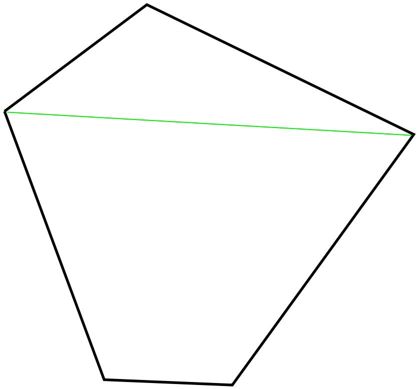

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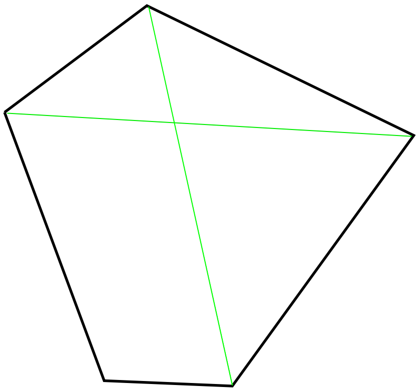

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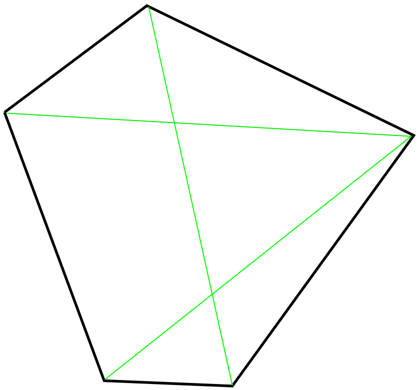

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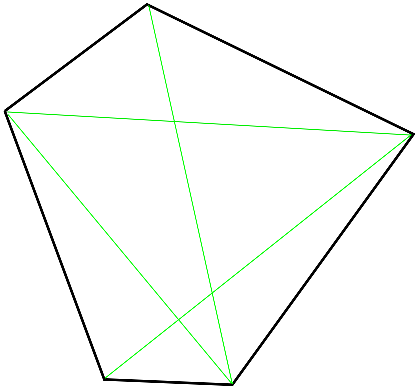

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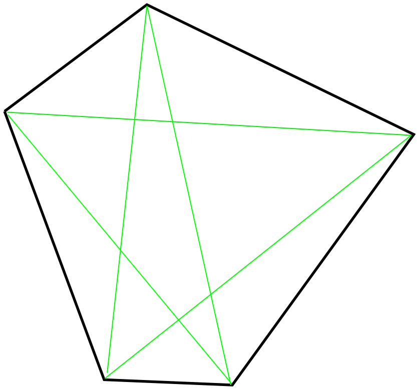

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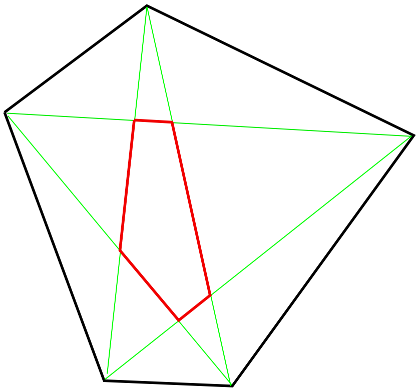

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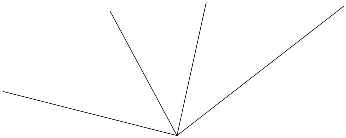

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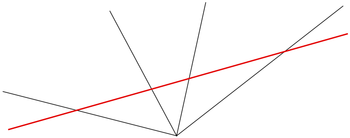

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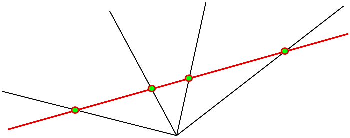

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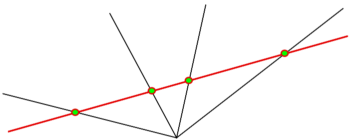

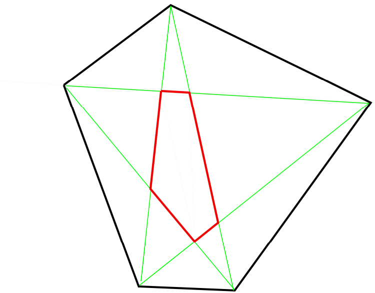

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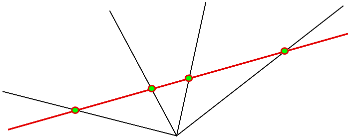

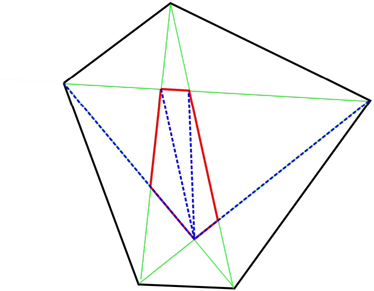

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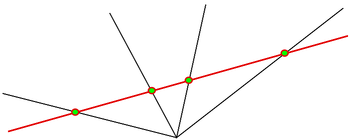

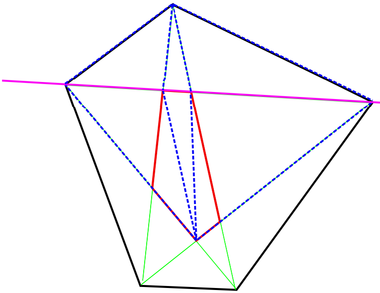

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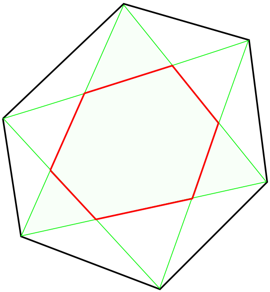

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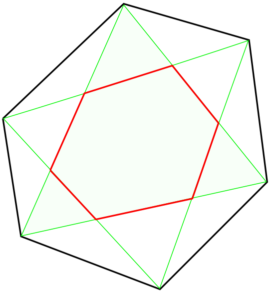

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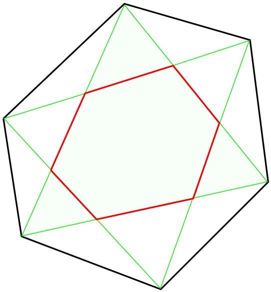

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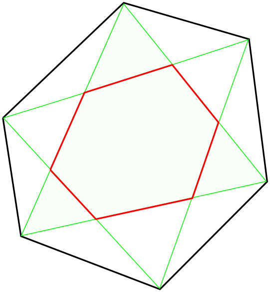

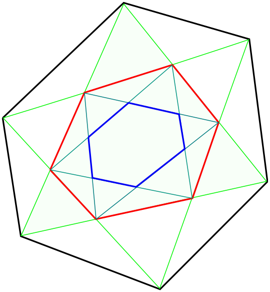

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The answer is: not necessarily, and it was only recently partially answered by Richard Schwartz, Serge Tabachnikov and Valentin Ovsienko in

The Pentagram map: a discrete integrable system,
Communications in Math. Physics (2010)

Definition

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Theorem

(Ovsienko, Schwartz, Tabachnikov 2010) The projective invariants of almost every universally convex n-gon lie on a smooth torus that has a $\tilde{T}_n$-invariant affine structure. That is, the invariants of almost every universally convex n-gon undergo quasi-periodic motion under the invariantization of the pentagram map.

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Theorem

(Ovsienko-Schwantz-Tabachnikov 2010) The continuous limit (in both time and space) of the invariantization of the pentagram map is the Boussinesq equation.

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- Configuration theorems: whether or not an n-gon is inscribed in a conic seems to be connected to whether or not its transformed by a sequence of pentagram-like maps is equivalent to itself.

For more information see the many papers by Schwartz et als in the subject.

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One needs to choose a guiding idea that will help narrow the possibilities.

Generalizations of Boussinesq equation - the AGD flows

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The Adler-Gel'fand-Dikii (AGD) flows are defined as follows (Adler 79): consider scalar differential operators of the form

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$$L_t = L(X_r L)_+ - (LX_r)_+ L$$

where $()_+$ selects the local terms, X_r is a Hamiltonian pseudo-differential operator defining the variation of the Hamiltonian

$$\mathcal{H}_r(L) = \int_{S^1} res(L^{r/(n+1)})dx$$

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The invariants k_i are called projective Wilczynski invariants for u.

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Let u be a parametrized curve in $\mathbb{RP}^m$ and let γ be a unique lift to $\mathbb{R}^{m+1}$ such that $\det(\gamma, \gamma', \dots, \gamma^{(m)}) = 1$.Then

$$\rho = (\gamma, \gamma', \dots, \gamma^{(m)})$$

is a left moving frame for u and the Maurer-Cartan matrix K is given by

$$\rho_x = \rho \begin{pmatrix} 0 & 0 & \dots & 0 & k_0 \\ 1 & 0 & \dots & 0 & k_1 \\ \ddots & \ddots & \dots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & k_{m-1} \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

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Theorem

(MB 00-06) There exists a geometric evolution of curves in $\mathbb{RP}^n$ inducing an r-AGD evolution on k_i, for all r. Furthermore, if an appropriate moving frame $\hat{\rho}$ is fixed, then the geometric evolution can be explicitly and algebraically found from $\hat{\rho}$ and $\delta\mathcal{H}_r$, where $\mathcal{H}_r$ is the r-AGD Hamiltonian.

Briefly:

We gauge the Wilczynski frame to a different frame $\hat{\rho}$ so that

$$\hat{\rho}_x = \hat{\rho} \begin{pmatrix} 0 & \kappa_m & \dots & \kappa_2 & \kappa_1 \\ 1 & 0 & \dots & 0 & 0 \\ \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

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One can also obtain a biHamiltonian structure on the κ_i induced by a general one existing on the space of loops on $\mathfrak{sl}(n+1)^*$, etc.

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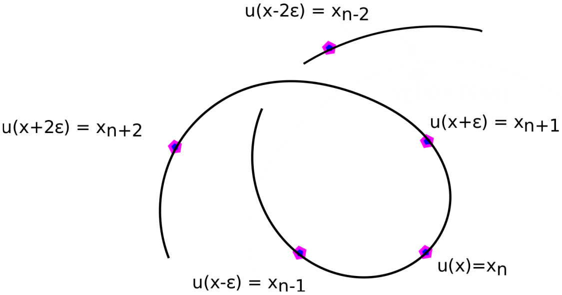

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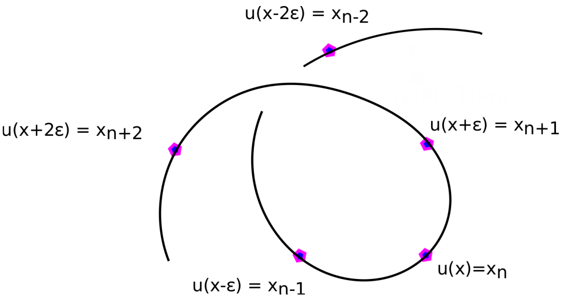

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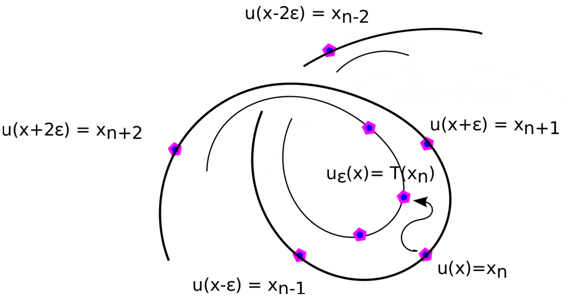

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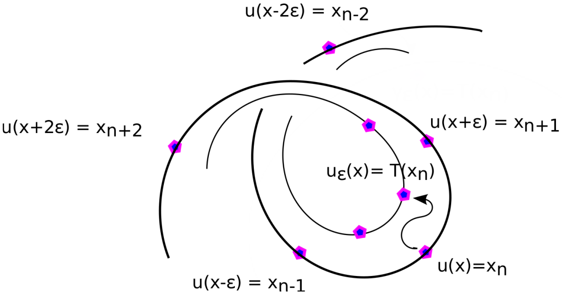

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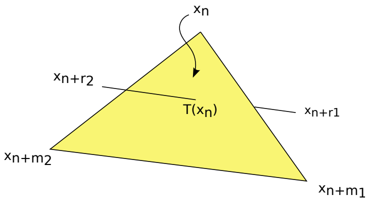

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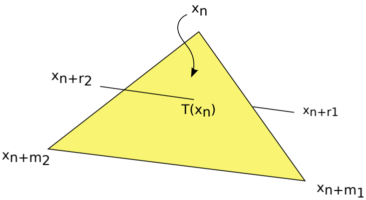

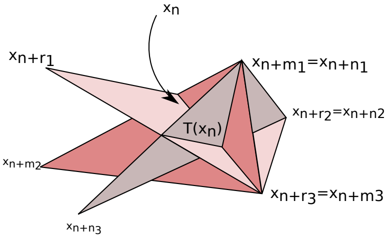

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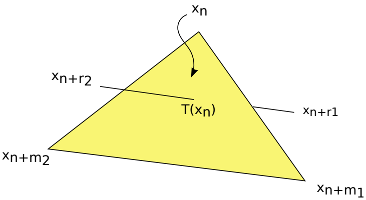

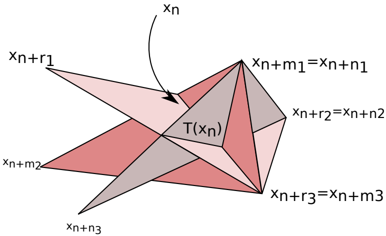

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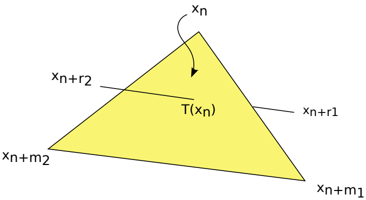

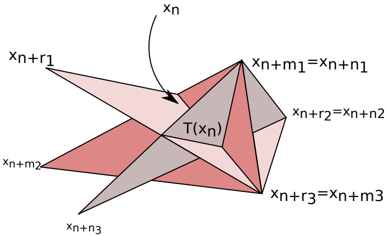

Notice that there are infinitely many choices of points. We can associate

$$\text{line/plane} \to (r_1, r_2, m_1, m_2) \qquad 3 \text{ planes} \to (m_i, n_i, r_i), \quad i = 1, 2, 3$$

and determine which integers will result in AGD limits.

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$$\gamma_\epsilon(x) = a_1\gamma(x + r_1\epsilon) + a_2\gamma(x + r_2\epsilon)$$

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There are infinitely many solutions, one of the simplest one is the three planes $\pi_1 = \langle x_{n-2}, x_{n+3}, x_{n+5}\rangle$, $\pi_2 = \langle x_{n-5}, x_{n+2}, x_{n+3}\rangle$ *and* $\pi_3 = \langle x_{n-5}, x_{n+1}, x_{n-6}\rangle$

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The AGD Hamiltonian flow associated to the $L^{\frac{k}{m+1}}$*–Hamiltonian is the continuous limit of maps defined analogously to the pentagram map through the intersection of one* $k-1$*-dimensional subspace and a group of* $k-1$ $m-1$*-dimensional subspaces of* $\mathbb{RP}^m$.

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MERCI!

THANKS!