KILLING TENSORS AND MOVING FRAMES

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We will discuss the study of Killing tensors (notably Killing two-tensors) defined in spaces of constant curvature via the method of moving frames.

Workshop on MOVING FRAMES IN GEOMETRY, CRM, 2011

Eisenhart L. P., *Separable systems of Stäckel*, Ann. Math **35**, 284-305 (1934).

Killing tensors

W. Killing (1892)

Let $(\mathcal{M}, \boldsymbol{g})$ be a pseudo-Riemannian manifold.

Dfn 1. *A* Killing tensor field $\boldsymbol{K}$ of valence $p \geq 1$ *defined in* $(\mathcal{M}, \boldsymbol{g})$ *is a symmetric* $(p, 0)$ *tensor field satisfying the Killing tensor equation*

$$[\boldsymbol{K}, \boldsymbol{g}] = 0, \tag{1}$$

where [,] denotes the Schouten bracket. When $p =$ 1, $\boldsymbol{K}$ *is said to be a Killing vector field (infinitesimal isometry) and (1) reduces to*

$$\mathcal{L}_{\boldsymbol{K}} \boldsymbol{g} = 0, \tag{2}$$

where $\mathcal{L}$ *denotes the Lie derivative operator.*

$\mathcal{K}^p(\mathcal{M})$ denotes the vector space of Killing tensors of valence p defined on $\mathcal{M}$. Its dimension derived independently (by Delong, Takeuchi, Thompson, and others) is given by

$$d = \dim \mathcal{K}^p(\mathcal{M}) \leq \frac{1}{n}\binom{n+p}{p+1}\binom{n+p-1}{p},$$

with equality iff $\mathcal{M}$ is of constant curvature.

Fore example, let $\boldsymbol{g}$ be the Minkowski metric

$$g_{ij} = \mathrm{diag}(-1, 1, 1),$$

defined with respect to canonical pseudo-Cartesian coordinates $x^i = (t, x, y)$.

In this coordinates the the vector space of Killing vectors is spanned by

$$\boldsymbol{X}_i = \frac{\partial}{\partial x^i}, \quad \boldsymbol{R}_i = \epsilon^k{}_{ji} x^j \boldsymbol{X}_k. \tag{3}$$

The general Killing tensor in $\mathcal{K}^2(\mathbb{M}^3)$ may be expressed as

$$\boldsymbol{K} = A^{ij} \boldsymbol{X}_i \odot \boldsymbol{X}_j + 2B^{ij} \boldsymbol{X}_i \odot \boldsymbol{R}_j + C^{ij} \boldsymbol{R}_i \odot \boldsymbol{R}_j, \tag{4}$$

where A^{ij}, B^{ij} and C^{ij}, the *Killing tensor parameters*, are constants and satisfy the symmetry properties $A^{ij} = A^{(ij)}$ and $C^{ij} = C^{(ij)}$. On account of the syzygy $g^{ij}\boldsymbol{X}_i \odot \boldsymbol{R}_j = 0$, only twenty of the twenty-one Killing tensor parameters are independent.

Of course, (4) is not the only available representation for $\mathcal{K}^2(\mathbb{M}^3)$. Alternatively, this vector space can be represented in terms of the components of an appropriate *algebraic curvature tensor*, which arises in this context quite naturally when one studies Killing two-tensors defined on spaces of constant non-zero curvature. For example, let $\mathcal{M} = \mathbb{S}^n$. Then,

$$\mathcal{K}^2(\mathbb{S}^n) = C^{ijk\ell}\boldsymbol{R}_{ij} \odot \boldsymbol{R}_{k\ell}, \tag{5}$$

where $\boldsymbol{R}_{ij} = 2\delta_{ij}^{k\ell} g_{\ell m} x^m \boldsymbol{X}_k, \quad \boldsymbol{X}_i = \frac{\partial}{\partial x^i},$

and x^i, $i = 1, \ldots, n+1$ are Cartesian coordinates of the corresponding ambient space $\mathbb{E}^{n+1}$.

The vector spaces of Killing tensors of valence two $\mathcal{K}^2(\mathcal{M})$ are of particular importance because (some of their) elements play a pivotal role in the Hamilton-Jacobi theory of orthogonal separation of variables.

Hamilton-Jacobi theory of orthogonal separation of variables

Given a Hamiltonian system defined by a natural Hamiltonian of the form

$$H(\boldsymbol{q}, \boldsymbol{p}) = \tfrac{1}{2} g^{ij}(\boldsymbol{q}) p_i p_j + V(\boldsymbol{q}). \tag{6}$$

We wish to know

(i) How many "inequivalent" coordinate systems afford orthogonal separation of variables in the corresponding Hamilton-Jacobi (HJ) equation

$$H(\boldsymbol{q}, \boldsymbol{p}) = E, \quad p_i = \frac{\partial W}{\partial q^i}, \quad i = 1, \ldots, n? \tag{7}$$

(ii) If the answer to (i) is non-zero, how can one characterize intrinsically the coordinate systems that afford separation of variables in the HJ equation?

(iii) What are the canonical coordinate transformations

$$(q^1, q^2, \ldots, q^n) \to (u^1, u^2, \ldots, u^n)$$

from the given position coordinates of (6) to the coordinate systems that afford orthogonal separation of variables of the HJ equation?

A natural connection with the Hamiltonian mechanics: a function $F \in T^*(\mathcal{M})$ which is quadratic in the momenta according to

$$F(\boldsymbol{q}, \boldsymbol{p}) = K^{ij}(\boldsymbol{q}) p_i p_j \tag{8}$$

is a *first integral* of (6) iff the functions K^{ij} above are the components of a Killing tensor field $\boldsymbol{K} \in \mathcal{K}^2(\mathcal{M})$.

Consider, for example, the Morosi-Tondo (MT) integrable system*

(9) $$H = \tfrac{1}{2}(2p_u p_v + {p_y}^2) - \tfrac{5}{8}u^4 + \tfrac{5}{2}u^2 v + \tfrac{1}{2}uy^2 - \tfrac{1}{2}v^2,$$

which is obtained as a stationary reduction of the seventh-order KdV flow. The Hamiltonian (9) is defined on the base manifold $\mathbb{M}^3$ with respect to the position-momenta coordinates $q^i = (u, v, y)$ and $p_i = (p_u, p_v, p_y)$, or in the pseudo-Cartesian coordinates t and x, $g_{ij} = \mathrm{diag}(-1, 1, 1)$ by

$$u = -\frac{1}{\sqrt{2}}(t + x), \quad v = \frac{1}{\sqrt{2}}(t - x),$$

Then, the potential in (9) assumes the form

(10) $$V = -\tfrac{5}{32}(t+x)^4 + \tfrac{5\sqrt{2}}{8}(t+x)^2(t-x) - \tfrac{\sqrt{2}}{4}(t+x)y^2 - \tfrac{1}{4}(t-x)^2.$$

*C. Morosi and G. Tondo, "Quasi-bi-Hamiltonian systems and separability", *J. Phys. A: Math. Gen.* **30**, 2799-2806 (1997).

Thm 2 (Eisenhart). *The Hamiltonian system (6) defined by the geodesic Hamiltonian ($V = 0$) is orthogonally separable iff it admits $n - 1$ functionally independent first integrals of motion of the form (8), such that (i) all of the corresponding Killing tensors of valence two have real and pointwise simple (almost everywhere) eigenvalues, (ii) the eigenvectors (or eigenforms) of these Killing two-tensors are normal and (iii) the Killing two-tensors defined by the $n-1$ first integrals have the same eigenvectors (eigenforms).*

Let $\boldsymbol{K}_1, \ldots, \boldsymbol{K}_{n-1}$ be the Killing two-tensors of Theorem 2. Then $\{\boldsymbol{g}, \boldsymbol{K}_1, \ldots, \boldsymbol{K}_{n-1}\}$ generates an n-dimensional vector subspace of $\mathcal{K}^2(\mathcal{M})$. The generic Killing tensor

$$\boldsymbol{K} = \boldsymbol{g} + \sum_{i=1}^{n-1} \boldsymbol{K}_i \tag{11}$$

has pointwise distinct eigenvalues and the same eigenvectors as any of the $\boldsymbol{K}_i$, $i = 1, \ldots, n-1$. The normality of the eigenvectors of each of the $n-1$ Killing tensors means that the eigenvectors generate n foliations that consist of $(n-1)$-dimensional hypersurfaces orthogonal to the eigenvectors of the Killing tensor. Such a geometric construction is called an *(orthogonal) separable web* which defines the coordinates of separation for the HJ equation (7).

The case of $V \neq 0$ in (6) is the subject of a more general theorem.

Thm 3 (Benenti)**.** *The natural Hamiltonian system defined by (6) is orthogonally separable iff there exists a valence-two Killing tensor* $\boldsymbol{K}$ *with (i) pointwise simple and real eigenvalues, (ii) normal eigenvectors (eigenforms) and (iii) such that*

$$\mathrm{d}(\hat{\boldsymbol{K}}\, \mathrm{d}V) = 0, \tag{12}$$

where the $(1,1)$*-tensor* $\hat{\boldsymbol{K}} = \boldsymbol{K}\boldsymbol{g}^{-1}$*.*

A Killing tensor satisfying conditions (i) and (ii) of Theorem 2 or 3 is called a *characteristic Killing tensor (CKT)*.

Therefore given a Killing two-tensor $\boldsymbol{K} \in \mathcal{K}^2(\mathcal{M})$, we wish to

1) verify whether or not it is a characteristic Killing tensor;

2) if it is, - then to determine what type of an orthogonal coordinate web it generates.

Or, in other words, we wish to classify the characteristic Killing tensors of $\mathcal{K}^2(\mathcal{M})$ modulo the action of the (orientation-preserving) isometry group G of $\mathcal{M}$, which leads to

1. **Canonical forms problem:** Consider the action $G \circlearrowright \mathcal{K}^2(\mathcal{M})$. The problem is to determine the number of *inequivalent* orbits corresponding to the CKTs defined on $(\mathcal{M}, \boldsymbol{g})$ as well as the canonical forms representing each of them.

2. **Equivalence problem:** Consider again the action $G \circlearrowright \mathcal{K}^2(\mathcal{M})$ (or $G \circlearrowright \mathcal{K}^2(\mathcal{M}) \times \mathcal{M}$). Let $\boldsymbol{K} \in \mathcal{K}^2(\mathcal{M})$. First, the problem is to determine whether or not $\boldsymbol{K}$ is a CKT. If the answer is "yes", the main problem is to determine the corresponding orbit in the quotient space $\mathcal{K}^2(\mathcal{M})/G$ (or $(\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G$) that the Killing two-tensor in question $\boldsymbol{K}$ belongs to. Finally, we also want to determine the *moving frames map* that maps $\boldsymbol{K}$ to its respective canonical form.

Let $\boldsymbol{K} \in \mathcal{K}^2(\mathcal{M})$ have real and distinct eigenvalues. To check whether its eigenvalues (eigenforms) are normal, one **cannot** use the vanishing of the Nijenhuis tensor:

$$N^i{}_{jk} = K^i{}_\ell K^\ell{}_{[j,k]} + K^\ell{}_{[j} K^i{}_{k],\ell} = 0. \tag{13}$$

Instead, one can use the Tonolo-Schouten-Nijenhuis conditions:

$$\begin{aligned} &N^\ell{}_{[jk} g_{i]\ell} = 0, \\ &N^\ell{}_{[jk} K_{i]\ell} = 0, \\ &N^\ell{}_{[jk} K_{i]m} K^m{}_\ell = 0, \end{aligned} \tag{14}$$

or the Haantjes condition:

$$H^i{}_{jk} = N^i{}_{\ell m} K^\ell{}_j K^m{}_k + 2N^\ell{}_{m[j} K^m{}_{k]} K^i{}_\ell + N^\ell{}_{jk} K^m{}_\ell K^k{}_m = 0. \tag{15}$$

Substituting the Killing tensor representation (5) into the condition (15) one gets

$$
\begin{aligned}
4C_{\ell(pq}{}^{k}C^{m}{}_{rs|i}C_{j|tu}{}^{n}C^{\ell}{}_{v)mn} &+ \\
2C_{\ell(p|m|}{}^{k}C^{n}{}_{qr[i}C_{j]st}{}^{m}C^{\ell}{}_{uv)n} &- \\
5C_{\ell(pq}{}^{k}C^{m}{}_{rs[i}C_{j]|m|t}{}^{n}C^{\ell}{}_{uv)n} &+ \\
C_{\ell(pq}{}^{k}C^{m}{}_{rs[i}C_{j]}{}^{\ell}{}_{t}{}^{n}C_{|n|uv)m} &+ \\
C_{\ell(pq}{}^{k}C^{m}{}_{rs[i}C_{j]tu}{}^{n}C_{|n|v)m}{}^{\ell} &- \\
3C_{\ell(pq}{}^{k}C^{m}{}_{r|ij|}C^{n}{}_{st|m|}C^{\ell}{}_{uv)n} &- \\
2C_{\ell(pq}{}^{k}C^{m}{}_{rs[i}C_{j]t|m|}{}^{n}C^{\ell}{}_{uv)n} &= 0.
\end{aligned}
\tag{16}
$$

Substituting the Killing tensor (5) into the Tonolo-Schouten-Nijenhuis condition (14), we get

$$C^{\ell}{}_{(pq[i}C_{jk]r)\ell} = 0, \tag{17}$$

$$\begin{aligned} &C_{\ell(pq}{}^{m}C^{\ell}{}_{r[ij}C_{k]st)m} - \\ &2C_{\ell(pq[i}C_{j}{}^{\ell}{}_{|r|}{}^{m}C_{k]st)m} = 0, \end{aligned} \tag{18}$$

$$\begin{aligned} &3C_{\ell(pq}{}^{m}C^{\ell}{}_{r|n|s}C^{n}{}_{t[ij}C_{k]uv)m} + \\ &2C_{\ell(pq}{}^{m}C_{|n|rs[i}C_{j|t|}{}^{n\ell}C_{k]uv)m} + \\ &2C_{\ell(pq}{}^{m}C_{|n|rs[i}C_{j}{}^{n}{}_{|t|}{}^{\ell}C_{k]uv)m} = 0, \end{aligned} \tag{19}$$

where $|\ldots|$ denotes exclusion of the enclosed indices from the symmetrization process.

Thm 4. [†] *[CSCMS] (17) + (18) $\Rightarrow$ (19).*

[†] Cochran (née Adlam) C. M., McLenaghan R. G., and Smirnov R. G., Equivalence problem for the orthogonal webs on the 3-sphere, JMP, 2011 (to appear)

Moving frames

In a fixed (quasi-)orthonormal frame of eigenvectors $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ the corresponding Cartan structure equations read

$$\mathrm{d}\boldsymbol{e}^a + \boldsymbol{\omega}^a{}_b \wedge \boldsymbol{e}^b = \boldsymbol{T}^a, \tag{20}$$

$$\mathrm{d}\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^c{}_b = \boldsymbol{\Theta}^a{}_b, \tag{21}$$

together with the Killing tensor equations for the components K_{ab} of $\boldsymbol{K}$,

$$K_{(ab;c)} = 0,$$

and the integrability conditions

$$\boldsymbol{e}^a \wedge \mathrm{d}\boldsymbol{e}^a = 0 \quad \text{(no sum)}.$$

In these equations, $\boldsymbol{\omega}^a{}_b = \Gamma_{cb}{}^a\,\boldsymbol{e}^c$ are the connection one-forms, $\boldsymbol{T}^a = \frac{1}{2}T^a{}_{bc}\,\boldsymbol{e}^b \wedge \boldsymbol{e}^c$ are the torsion two-forms, $\boldsymbol{\Theta}^a{}_b = \frac{1}{2}R^a{}_{bcd}\,\boldsymbol{e}^c \wedge \boldsymbol{e}^d$ are the curvature two-forms, $\{\boldsymbol{e}^1, \dots, \boldsymbol{e}^n\}$ is the dual basis of one-forms, the connection coefficients $\Gamma_{cb}{}^a$ correspond to the Levi-Civita connection ∇ (and hence $\boldsymbol{T}^a = 0$ in (20)) and $R^a{}_{bcd}$ are the components of the curvature tensor. We also note that, with respect to this frame, the components of the metric $\boldsymbol{g}$ and CKT $\boldsymbol{K}$ are given by

$$g_{ab} = \mathrm{diag}(\epsilon_1, \dots, \epsilon_n), K_{ab} = \mathrm{diag}(\epsilon_1\lambda_1, \dots, \epsilon_n\lambda_n),$$

respectively, where $\epsilon_a = \pm 1$, $a = 1, \dots, n$, and λ_a, $a = 1, \dots, n$, are the eigenvalues of $\boldsymbol{K}$.

$$\mathbf{e}^a = f_a du^a, \quad \mathbf{e}_a = \frac{1}{f_a}\frac{\partial}{\partial u^a}.$$

Eisenhart (1934) solved the canonical forms problem for $\mathcal{K}^2(\mathbb{E}^3)$, by reducing the Killing tensor equation to

$$\frac{\partial \lambda_a}{\partial u^a} = 0, \quad \frac{\partial \lambda_a}{\partial u^b} = (\lambda_a - \lambda_b)\frac{\partial \ln f_a^2}{\partial u^b}$$

and deriving from the above equations integrability conditions upon demanding that the eigenvalues λ_a be distinct and that their mixed second-order partial derivatives commute, thus proving that there were exactly 11 orbits corresponding to characteristic Killing tensors of the space $\mathcal{K}^2(\mathbb{E}^3)$.

Or, more geometrically, the problems can be described as follows:

$$
\begin{tikzcd}
G \arrow[r, "\pi_1"] & G/H \simeq \mathcal{M} \\
(\mathcal{K}^2(\mathcal{M}) \times \mathcal{M})/G \arrow[u, "f"] & \mathcal{K}^2(\mathcal{M}) \times \mathcal{M} \arrow[l, "\pi_3"'] \arrow[u, "\pi_2"']
\end{tikzcd}
\tag{22}
$$

Solving the canonical forms problem, one gets a set of webs, along with the corresponding metrics and coordinate systems. Here is the spherical case for $\mathcal{K}^2(\mathbb{S}^3)$:

$$\begin{cases} \mathrm{d}s^2 = \mathrm{d}t^2 + \sin^2 t(\mathrm{d}u^2 + \sin^2 u \mathrm{d}v^2) \\ x = \sin t \sin u \cos v, \;\; y = \sin t \sin u \sin v, \\ z = \sin t \cos u, \;\; w = \cos t \\ 0 \leqslant t \leqslant \pi, \;\; 0 \leqslant u \leqslant \pi, \;\; 0 \leqslant v < 2\pi \end{cases}$$

$$\boldsymbol{K} = c_1 R_{12} \odot R_{12} + c_2(R_{13} \odot R_{13} + R_{23} \odot R_{23})$$

$\boldsymbol{R}_{ij} = \mathrm{diag}(c_1 + c_2,\; c_1 + c_2,\; 2c_2, 0)$ (the corresponding canonical Ricci tensor)

Because the isometry group acts transitively in the bundle of frames one can consider instead the actual group action and the algebraic moving frames method (Fels Olver 1997-98). For example,

The action of $\mathsf{SE}(2,1) \circlearrowright \mathcal{K}^2(\mathbb{M}^3) \times \mathbb{M}^3$ is given by

$$x^i = \Lambda^i{}_j \tilde{x}^j + \delta^i, \tag{23}$$

where $\Lambda^i{}_j \in \mathsf{SO}(2,1)$, $\delta^i \in \mathbb{R}^3$, and

$$\begin{aligned} \tilde{A}^{ij} &= \Lambda_k{}^i \Lambda_\ell{}^j A^{k\ell} + 2\Lambda_k{}^{(i} \mu_\ell{}^{j)} B^{k\ell} + \mu_k{}^i \mu_\ell{}^j C^{k\ell}, \\ \tilde{B}^{ij} &= \Lambda_k{}^i \Lambda_\ell{}^j B^{k\ell} + \mu_k{}^i \Lambda_\ell{}^j C^{k\ell}, \\ \tilde{C}^{ij} &= \Lambda_k{}^i \Lambda_\ell{}^j C^{k\ell}. \end{aligned} \tag{24}$$

where

$$\mu_i{}^j = \epsilon^k{}_{\ell i} \Lambda_k{}^j \delta^\ell. \tag{25}$$

To generate SE$(2,1)$-covariants of $\mathcal{K}^2(\mathbb{M}^3)$, we define

$$K^{ij} = A^{ij} + 2\epsilon^{(i}{}_{\ell k} B^{j)k} x^\ell + \epsilon^i{}_{mk} \epsilon^j{}_{n\ell} C^{k\ell} x^m x^n, \tag{26}$$

$$L^{ij} = B^{ij} + \epsilon^i{}_{\ell k} C^{jk} x^\ell, \tag{27}$$

Some of the requisite covariants:

$$\begin{aligned}
&\mathcal{C}_1 = \mathsf{Tr}(\boldsymbol{C}), \quad \mathcal{C}_2 = \mathsf{Tr}(\boldsymbol{C}^2), \quad \mathcal{C}_3 = \mathsf{Tr}(\boldsymbol{C}^3), \\
&\mathcal{C}_4 = \mathsf{Tr}(\boldsymbol{L}^2), \quad \mathcal{C}_5 = \mathsf{Tr}(\boldsymbol{L}\boldsymbol{L}^t), \quad \mathcal{C}_6 = \mathsf{Tr}(\boldsymbol{L}\boldsymbol{C}\boldsymbol{L}^t), \\
&\mathcal{C}_7 = \mathsf{Tr}(\boldsymbol{L}\boldsymbol{C}^2\boldsymbol{L}), \quad \mathcal{C}_8 = \mathsf{Tr}(\boldsymbol{K}\boldsymbol{C}), \quad \mathcal{C}_9 = \mathsf{Tr}(\boldsymbol{K}\boldsymbol{C}\boldsymbol{K}\boldsymbol{C}), \\
&\mathcal{C}_{10} = \mathsf{Tr}(\boldsymbol{K}\boldsymbol{L}\boldsymbol{K}\boldsymbol{L}^t), \quad \mathcal{C}_{11} = \mathsf{Tr}(\boldsymbol{K}\boldsymbol{L}^2), \quad \mathcal{C}_{12} = \mathsf{Tr}(\boldsymbol{K}^2\boldsymbol{L}^2).
\end{aligned} \tag{28}$$

Solving the equivalence problem

To separate the orbits, we use invariants, covariants, isotropy subgroups.

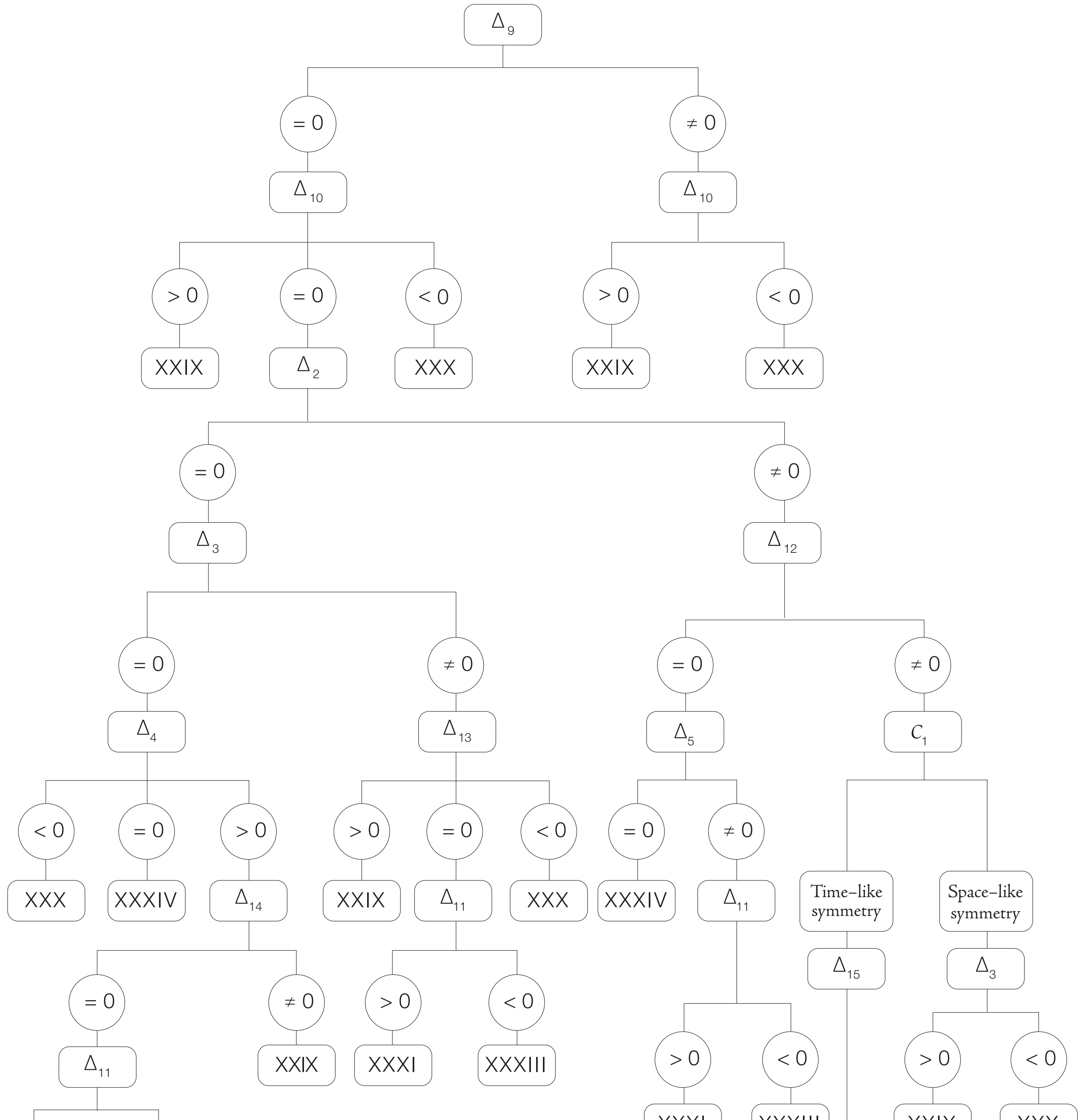
$$\dim G = \dim O_{\mathbf{x}} + \dim G_{\mathbf{x}}$$

Computing the corresponding *moving frames map* can be hard. If it cannot be found directly from the characteristic Killing tensor $\boldsymbol{K}$ in question, use the infinitesimal generators of the isotropy subgroup (map them into their respective canonical forms instead), the corresponding Ricci tensor derived from the algebraic curvature tensor, its eigenvalues.

The $\mathcal{K}^2(\mathbb{H}^3)$ case

Caroline Cochran (née Adlam), "The equivalence problem for orthogonally separable webs on spaces of constant curvature", Dalhousie University, 2011.

$\mathcal{K}^2(\mathbb{H}^3)$ $\mathcal{K}^2(\mathbb{S}^3)$



Δ_9
$= 0$
$\neq 0$
Δ_{10}
Δ_{10}
> 0
$= 0$
< 0
> 0
< 0
XXIX
Δ_2
XXX
XXIX
XXX
$= 0$
$\neq 0$
Δ_3
Δ_{12}
$= 0$
$\neq 0$
$= 0$
$\neq 0$
Δ_4
Δ_{13}
Δ_5
C_1
< 0
$= 0$
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XXX
XXXIV
Δ_{14}
XXIX
Δ_{11}
XXX
XXXIV
Δ_{11}
Time–like symmetry
Space–like symmetry
Δ_{15}
Δ_3
$= 0$
$\neq 0$
> 0
< 0
> 0
< 0
> 0
< 0
Δ_{11}
XXIX
XXXI
XXXIII

Solution

Impose the compatibility condition $\mathrm{d}(\hat{\boldsymbol{K}}\mathrm{d}V) = 0$ on the potential V (10), solve the system of PDEs:

(29) $$K^{ij} = a_1 K_1^{ij} + a_2 K_2^{ij} + a_3 K_3^{ij},$$

where a_1, a_2 and a_3 are arbitrary constants and

$$K_1^{ij} = \begin{pmatrix} 1+2\sqrt{2}\,x & 1+\sqrt{2}(t-x) & -\sqrt{2}\,y \\ 1+\sqrt{2}(t-x) & 1-2\sqrt{2}\,t & \sqrt{2}\,y \\ -\sqrt{2}\,y & \sqrt{2}\,y & -2\sqrt{2}(t+x) \end{pmatrix},$$

$$K_2^{ij} = \begin{pmatrix} -2-2\sqrt{2}\,x & -1-\sqrt{2}(t-x) & \sqrt{2}\,y \\ -1-\sqrt{2}(t-x) & 2\sqrt{2}\,t & -\sqrt{2}\,y \\ \sqrt{2}\,y & -\sqrt{2}\,y & 1+2\sqrt{2}(t+x) \end{pmatrix},$$

$$K_3^{ij} = \begin{pmatrix} -y^2 & y^2 & -y(t+x+\sqrt{2}) \\ y^2 & -y^2 & y(t+x-\sqrt{2}) \\ -y(t+x+\sqrt{2}) & y(t+x-\sqrt{2}) & -(t+x)^2 - 2\sqrt{2}(t-x) \end{pmatrix}.$$

The TSN conditions (14) are identically satisfied for the Killing tensor (29) and hence it has normal eigenvectors for all a_1, a_2 and a_3. The discriminant of the characteristic polynomial of (29) is a lengthy polynomial in the constants a_i and the pseudo-Cartesian coordinates, nevertheless it is generally non-zero and vanishes only if $a_1 = a_2$ and $a_3 = 0$, in which case (29) reduces to a multiple of the metric. Therefore, we conclude that (29) generally has normal eigenvectors and real and distinct eigenvalues, thereby defining a characteristic Killing tensor (CKT).

The CKT (29) admits no web symmetry for any values of the constants a_i. The search for a dilatational web symmetry proves equally unsuccessful. Therefore, the CKT (29) characterizes one of the ten asymmetric separable webs in $\mathbb{M}^3$.

We now proceed to classify the asymmetric Killing tensor (29). It follows that $\mathcal{A}_1 = 0$, $\mathcal{C}_2 = 0$ and

$$\mathcal{A}_2 = 4\sqrt{2}\, {a_3}^2(t + x).$$

There are two cases to consider, namely $a_3 = 0$ and $a_3 \neq 0$. Firstly, if $a_3 = 0$, then $\mathcal{A}_4 = 0$ and $\mathcal{A}_{11} = -8(a_1 - a_2)^3 \neq 0$ (otherwise the CKT would reduce to the metric). Therefore, the CKT (29) with $a_3 = 0$ characterizes the asymmetric web II. Secondly, if $a_3 \neq 0$, then $\mathcal{A}_4 = 0$ and $\mathcal{A}_6 = -8{a_3}^3$, thus in this case the CKT also characterizes the asymmetric web II.

We now compute the moving frame map for (29) which transforms it to the corresponding canonical form.

$$\Lambda^i{}_j = \frac{1}{2\sqrt{2}} \begin{pmatrix} -3 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & -2\sqrt{2} \end{pmatrix}, \qquad \delta^i = 0, \tag{30}$$

defines the moving frame map. Similarly, for the case $a_3 \neq 0$, we find that the moving frame map for this case is also given by (30).

The transformation from canonical pseudo-Cartesian coordinates (t, x, y) to separable coordinates (μ, ν, ω) is given by

$$
\begin{aligned}
t + x &= \tfrac{1}{8}\big(\omega^2 + (\mu+\nu)^2\big)\big(\omega^2 + (\mu-\nu)^2\big), \\
t - x &= \mu^2 + \nu^2 - \omega^2, \\
y &= \mu\nu\omega.
\end{aligned}
\tag{31}
$$

The transformation to the separable coordinates is given by

$$
\begin{aligned}
u &= -\tfrac{1}{\sqrt{2}}(t + x) = \tfrac{1}{2}(\mu^2 + \nu^2 - \omega^2), \\
v &= \tfrac{1}{\sqrt{2}}(t - x) = -\tfrac{1}{8}\big(\omega^2 + (\mu+\nu)^2\big)\big(\omega^2 + (\mu-\nu)^2\big), \\
y &= -\mu\nu\omega.
\end{aligned}
\tag{32}
$$

Separation of variables on 3D-spaces of constant curvature

1. $\mathcal{K}^2(\mathbb{E}^3)$ - 11 orthogonal webs

   Canonical forms problem

   (a) Eisenhart, L. P., "Separable systems of Stäckel," Ann. Math. **35**, 284–305 (1934).

   Equivalence problem

   (a) Horwood, J. T., McLenaghan, R. G., and Smirnov, R. G., "Invariant classification of orthogonally separable Hamiltonian systems in Euclidean space," Commun. Math. Phys. **259**, 679–709 (2005).

   (b) Horwood, J. T., "On the theory of algebraic invariants of vector spaces of Killing tensors," J. Geom. Phys. **58**, 487–501 (2008).

2. $\mathcal{K}^2(\mathbb{M}^3)$ - 39 orthogonal webs (59 coordinate systems)

<u>Canonical forms problem</u>

(a) Horwood, J. T., and McLenaghan, R. G., "Orthogonal separation of variables for the Hamilton-Jacobi and wave equations in three-dimensional Minkowski space," J. Math. Phys. **49**, 023501 (48 pages) (2008).

<u>Equivalence problem</u>

(a) Horwood, J. T., McLenaghan, R. G., and Smirnov, R. G., Hamilton-Jacobi theory in three-dimensional Minkowski space via Cartan geometry, J. Math. Phys. **50**, 053507 (2009) (41 pages).

3. $\mathcal{K}^2(\mathbb{S}^3)$ - 5 orthogonal webs (6 coordinate systems)

Canonical forms problem

(a) Eisenhart, L. P., "Separable systems of Stäckel," Ann. Math. **35**, 284–305 (1934).

(b) Olevsky, M. N, "Three orthogonal systems in spaces of constant curvature in which the equation $\Delta_2 u + \lambda u = 0$ admits a complete separation of variables," Math. Sbornik **27**, 379–426 (1950).

Equivalence problem

(a) Cochran (née Adlam) C. M., McLenaghan R. G., and Smirnov R. G., Equivalence problem for the orthogonal webs on the 3-sphere, JMP, 2011 (to appear).

4. $\mathcal{K}^2(\mathbb{H}^3)$ - 29 orthogonal webs (34 coordinate systems)

Canonical forms problem

(a) Olevsky, M. N., "Three orthogonal systems in spaces of constant curvature in which the equation $\Delta_2 u + \lambda u = 0$ admits a complete separation of variables," Math. Sbornik **27**, 379–426 (1950).

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