Exterior Differential Systems, from Elementary to Advanced

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Abstract. These notes were developed for lectures at the Institute of Mathematics at the Polish Academy of Sciences in September 2016.

Exterior differential systems have been applied with great success to many problems in geometry and analysis. But, the techniques have changed very little since Cartan, where results are explained by deft manipulation of differential forms. In fact, differential forms are *not* the central idea in the theory of exterior differential systems; rather, the central idea is that one can solve differential equations by studying the geometry of the initial-value problem on particular linear subspaces called integral elements. Differential forms are merely a computational means to this goal.

In studying the geometry of initial-value problems, some under-appreciated advances have been made in the past 50 years by studying the parametrization of solutions via the characteristic variety, Ξ. Unfortunately, a lack of introductory resources is a barrier to entry for individual scholars, and I believe it has been a major impediment to progress in the field.

The goal of these lectures is to lower this barrier by exposing the audience to simplified versions of several key results regarding the characteristic variety, and to outline how these results could be used to push the frontiers of the field. These key results are:

(i) The incidence correspondence of the characteristic variety
(ii) Guillemin normal form and Quillen's thesis
(iii) The Integrability of Characteristics (Guillemin, Quillen, Sternberg, Gabber)
(iv) Yang's Hyperbolicity Criterion

The approach is *elementary*—a double entendre: First, elementary means that we rely on techniques that should be accessible to early graduate students. The notation might become complicated, but all results involve explicit matrix arithmetic. We allow ourselves to choose temporary bases for direct computation of homomorphisms, but we are very careful to avoid splitting short-exact sequences that might confuse isomorphic-but-distinct spaces. Second, elementary means that we focus on the geometry over integral elements. The story begins not with wedges and hooks, but with the geometry of the Grassmann manifold and its tautological bundle.

The first "elementary" is important, because it lowers the barrier to explicit construction and allows exploration with computer algebra systems. The second "elementary" is important, because it guarantees invariance of our results under local diffeomorphism without additional work.

The required background for these lectures is graduate-level linear algebra (short-exact sequences, dual spaces, the rank-nullity theorem, tensor products, generalized eigenspaces, as in Artin's *Algebra* [**Art91**]), the fundamentals of smooth manifolds (tangent spaces, Sard's theorem, bundles, as in Milnor's *Topology from the Differential Viewpoint* [**Mil97**]) and basic algebraic geometry (projective space, ideal, variety, scheme, as in Harris' *Algebraic Geometry, a first course* [**Har92**]). These lectures assume that the audience has a general cultural awareness of PDE or EDS in some form but the required definitions are provided.

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0. Introduction

Given a system of PDEs, does it have any local solutions to the free Cauchy problem? If so, how many? What is the shape of the family of local solutions? How can we determine whether two systems of PDEs are "the same" up to local coordinate transformations? Does the space of all PDEs (up to local coordinate transformation) have any meaningful shape of its own?

These questions are more geometric than analytic, and it is not surprising that the language of ideals, varieties, moduli, bundles, and schemes come into play. The following sections in Part I fix notation and remind the reader of the basic structures necessary to study EDS.

Section 1 introduces tableaux, from the perspective of basic linear algebra that would be familiar (in some notation or other) to any undergraduate math major. The promise of EDS is that our motivating questions regarding PDEs can be answered via detailed consideration of the geometry of tableaux. Sections 2 and 3 build the bridge between tableaux and PDEs. Section 2 introduces the Grassmann manifold as the geometry of linear subspaces, with emphasis on tangents and intersections. This should be accessible to any graduate student who has had an introductory course on manifolds or in classical algebraic geometry. Finally, Section 3 introduces Exterior Differential Systems as a way to describe varieties in the Grassmann bundle over a manifold, which is our favorite interpretation of the term "Partial Differential Equation." Together, these lay the groundwork for the results in Parts II and III, which are the purpose of this course.

Everything in these pages can be found in numerous places in the literature, and I have indicated my favorite sources throughout. As always, it is wise to have Bryant, et al.'s *Exterior Differential Systems* [**BCG**$^+$**90**] and Ivey and Landsberg's *Cartan for Beginners* [**IL03**] nearby.

The only innovations here are in presentation:

(i) The central topic is the C^∞ characteristic variety, not the C^ω Cartan–Kähler theorem. This is because I am interested in the question "what does the family of all PDEs look like?" not "how do I solve this particular PDE?"

(ii) Guillemin normal form plays the central computational role, not differential forms. This is because most researchers outside the field—and their computer algebra systems—are more comfortable with matrices than with exterior algebra.

(iii) Exterior differential ideals are not introduced until absolutely needed. This is because many of the essential lemmas depend only on the geometry of the Grassmann variety, which is the variety of the trivial exterior differential system.

For readability, many proofs are omitted or reduced to discussion in prose. This should not be an impediment to understanding — most of the proofs are basic linear algebra (in fact, almost all the proofs are the restatements of the rank-nullity theorem), and details are provided in the references.

Part I

Background Concepts

1. Tableaux and Symbols

Given vector spaces or projective spaces W and V, a *tableau* is a linear subspace of $A \subset \mathrm{Hom}(V,W)$. Tableaux are very simple objects—every undergraduate encounters the example "$r \times n$ matrices form a vector space using the usual matrix operations"—and a tableau is any subspace of that vector space. We use the notation $W \otimes V^*$ and $\mathrm{Hom}(V,W)$ interchangeably, and eventually we switch from vector spaces to complex projective spaces for algebraic convenience.

A tableau is the kernel of some linear map σ, called the *symbol*, whose range is written as $H^1(A)$. We have a short exact sequence of spaces:

$$0 \to A \to W \otimes V^* \xrightarrow{\sigma} H^1(A) \to 0, \tag{1.1}$$

where $H^1(A)$ is just notation for $(W \otimes V^*)/A$. Let $\dim A = s$ and $\dim H^1(A) = t = nr - s$.

For example, let $W = \mathbb{R}^3$ and $V = \mathbb{R}^3$, and consider the tableau A described by

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix}. \tag{1.2}$$

The symbol σ consists of four conditions on a 3×3 matrix (π^a_i):

$$\begin{aligned} 0 &= \pi^2_3 - \pi^3_2, \\ 0 &= \pi^1_3 - \pi^3_1, \\ 0 &= \pi^2_2 - \pi^3_1, \\ 0 &= \pi^1_2 - \pi^2_1 \end{aligned} \tag{1.3}$$

1(a). Rank-One Ideal. The fundamental theorem of linear algebra states that any homomorphism $\pi \in W \otimes V^*$ has a well-defined rank. Thus, for any tableau $A \subset W \otimes V^*$, we could ask how $\operatorname{rank} \pi$ varies across $\pi \in A$. For our purposes, the most important case is $\operatorname{rank} \pi = 1$.

The space $W \otimes V^*$ admits the *Rank-One Ideal*, $\mathscr{R}$, which is irreducible and generated by all 2×2 minors $\left\{0 = \pi^a_i \pi^b_j - \pi^a_j \pi^b_i\right\}$ in any basis. This is a homogeneous ideal, so we may consider the rank-one cone in vector space or the rank-one variety in projective space. For any A, we define $\mathscr{C} \subset A$ as the variety $\mathscr{C} = A \cap \mathrm{Var}(\mathscr{R})$—the set of matrices in A that are also rank-one—defined by the ideal $A^\perp + \mathscr{R}$.

In the example (1.2), $\mathscr{C}$ can be parametrized as matrices of the form

$$\begin{pmatrix} \lambda^4 & \lambda^3\tau & \lambda^2\tau^2 \\ \lambda^3\tau & \lambda^2\tau^2 & \lambda\tau^3 \\ \lambda^2\tau^2 & \lambda\tau^3 & \tau^4 \end{pmatrix} = \begin{pmatrix} \lambda^2 \\ \lambda\tau \\ \tau^2 \end{pmatrix} \otimes \begin{pmatrix} \lambda^2 & \lambda\tau & \tau^2 \end{pmatrix}, \tag{1.4}$$

which can be interpreted as the rational normal Veronese curve[1],

$$[1 : \tau : \tau^2 : \tau^3 : \tau^4] \subset \mathbb{P}^4 \cong \mathbb{P}A. \tag{1.5}$$

[1] For more on Veronese curves and the related Segre embeddings, see [**Har92**, **Sha94**].

Moreover, the projection of $\mathscr{C}$ to $\mathbb{P}V^*$ is another rational normal curve,

$$[1 : \tau : \tau^2] \subset \mathbb{P}^2 \cong \mathbb{P}V^*. \tag{1.6}$$

This toy example plays a crucial role in applications for hyperbolic and hydrodynamically integrable PDEs [**FHK09**, **Smi09**].

1(b). Generic Bases. For a particular homomorphism $B : \mathbb{C}^r \to \mathbb{C}^p$, there are various "good" bases to express B; when B is written in a "good" basis, we say it is in a normal form. The first example is to choose a basis of W via Gaussian elimination[2] such that B is in row-echelon form. A more sophisticated option is the singular-value decomposition, which requires bases of both $\mathbb{C}^{r*}$ and $\mathbb{C}^p$. In the case $p = r$, the best option is usually Jordan normal form, which arises by solving the generalized eigenspace problem.

Given a tableau $A \subset W \otimes V^*$, we are curious whether we can construct bases that are "good" simultaneously for all homomorphisms in the tableau. This situation is considerably more complicated than the situation of a single homomorphism, but we arrive at a satisfying answer in Section 5. Here is the first step:

In any bases of V^* and W, the tableau A is a space of $r \times n$ matrices only s of whose entries are linearly independent. That is, in a given basis, we can consider the entries $\pi \mapsto \pi^a_i$ as elements of V^*, just as we think of the component v^1 as being a linear function on $v \in \mathbb{R}^n$, using some basis. Across all bases of V^*, there is a maximum number of independent entries that can occur in column 1; call that number s_1. (In a measure-zero set of bases of V^*, the number of actual independent entries in the first column may may be less than s_1.) Once those independent entries are accounted for, there is a maximum number s_2 of new independent entries that can occur in the second column. (In a measure-zero set of bases of V^* that achieve s_1 in column 1, the number of actual independent entries in the columns 1 and 2 may be less than $s_1 + s_2$.) Continuing in this way, we have s_i as the number of new independent entries in the ith column achieved for almost-all bases of V^*. Note that $s_1 \geq s_2 \geq \cdots \geq s_i$, since otherwise the maximality condition would have been violated in an earlier column.

Eventually, we have reached $s_1 + s_2 + \cdots = s$, so there is some maximum column $\ell \leq n$ such that $s_\ell > 0$, where the last generator appears. So,

$$\begin{aligned} s &= s_1 + s_2 + \cdots + s_\ell + s_{\ell+1} + \cdots + s_n \\ &= s_1 + s_2 + \cdots + s_\ell + 0 + \cdots + 0. \end{aligned} \tag{1.7}$$

The index ℓ is called the *character* of A, and the number s_ℓ is called the *Cartan integer* of A. The tuple $(s_1, \ldots, s_\ell)$ gives the *Cartan characters* of A.

[2] Algorithmically, this is usually accomplished using improved Gram-Schmidt or Householder triangularization. See [**TB97**].

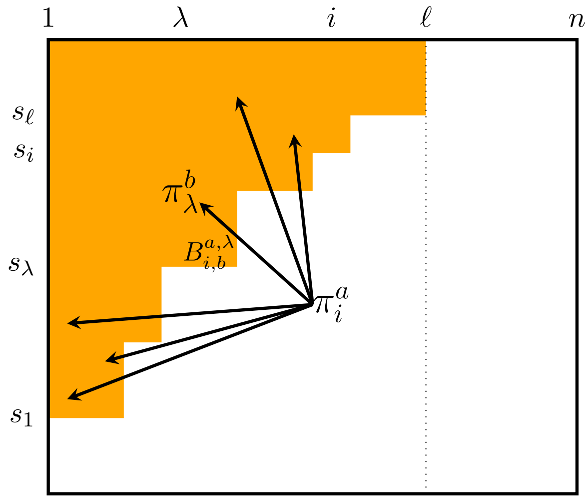

FIGURE 1. A tableau in coordinates, from [Smi15].

Permanently reserve the index ranges

$$\begin{aligned} \lambda, \mu &\in \{1, \ldots, \ell\} \\ \varrho, \varsigma &\in \{\ell + 1, \ldots, n\} \\ i, j &\in \{1, \ldots, n\} \\ a, b &\in \{1, \ldots, r\} \end{aligned} \tag{1.8}$$

A basis[3] $(u^i) = (u^1, \ldots, u^n)$ of V^* is called *generic* if its characters achieve the lexicographical maximum value $(s_1, s_2, \ldots, s_n)$. Almost all bases of V^* are generic. Given a basis (u^i) of V^*, a basis $(z_a) = (z_1, \ldots, z_r)$ is called *generic* if the *first* s_i independent entries in column i are independent.

Choose generic a basis $(u^i) = (u^1, \ldots u^n)$ for V^*, and let $(u_i) = (u_1, \ldots, u_n)$ be its dual basis for V. Choose a generic basis $(z_a) = (z_1, \ldots, z_r)$ for W, and let $(z^a) = (z^1, \ldots, z^n)$ be its dual basis for W^*. An element of the tableau is written as $\pi = \pi^a_i (z_a \otimes u^i) \in W \otimes V^*$. Because the bases are generic, the symbol map σ can be written as

$$\left\{ 0 = \pi^a_i - B^{a,\lambda}_{i,b} \pi^b_\lambda \right\}_{s_i < a}. \tag{1.9}$$

(Compare to the example (1.3), which is not written in generic bases. If you exchange columns $2 \leftrightarrow 3$ and rows $1 \leftrightarrow 3$, then it becomes generic.) It is implicit that $B^{a\lambda}_{i,b} = 0$ for $a \leq s_i$ and for $b \geq s_\lambda$ and for $i < \lambda$. That is, entries to the lower-right are written as linear combinations of the entries in the upper-left using the coefficients $B^{a,\lambda}_{i,b}$, as in Figure 1.

Another way to interpret the symbol coefficients $B^{a,\lambda}_{i,b}$ is as a map from the generating entries to the other entries. That is, consider the map

[3] This notation indicates an ordered basis, not a vector. Each u^i is an element of V^*.

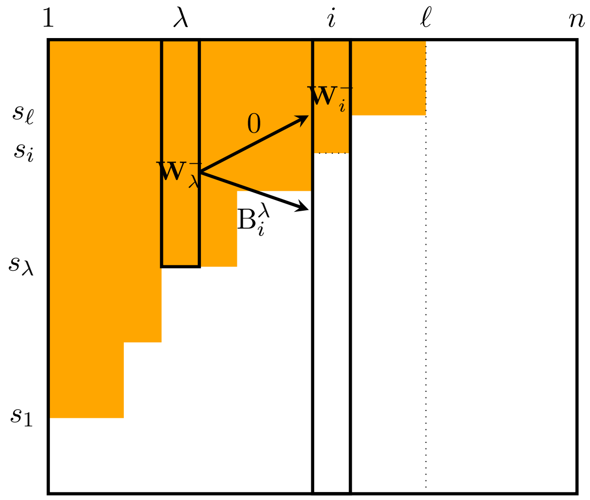

FIGURE 2. The map B_i^λ for a tableau in generic bases.

$\mathrm{B} \in V^* \otimes V \otimes W \otimes W^* \cong \mathrm{End}(V^*) \otimes \mathrm{End}(W)$ defined by

$$\sum_{a \le s_i} \delta_i^\lambda \delta_b^a (z_a \otimes z^b) \otimes (u^i \otimes u_\lambda) + \sum_{a > s_i} B_{i,b}^{a,\lambda} (z_a \otimes z^b) \otimes (u^i \otimes u_\lambda). \tag{1.10}$$

Equation (1.10) is the formal inclusion $A \to W \otimes V^*$ in the defining exact sequence (1.1).

By fixing $\varphi \in V^*$ and $v \in V$, we obtain an endomorphism $\mathrm{B}(\varphi)(v) : W \to W$ defined by the column relations of (π_i^a), as in Figure 2. We use the shorthand B_i^λ for $\mathrm{B}(u^\lambda)(u_i)$, but note that this is not quite the same as $B_{i,b}^{a,\lambda} z_a \otimes z^b$ because of the identity term in Equation (1.10).

Let U denote the subspace of V spanned by $u_1, \ldots, u_\ell$, so that $U^\perp$ is spanned by $u^{\ell+1}, \ldots, u^n \in U^*$. Let Y denote the subspace of V spanned by $u_{\ell+1}, \ldots, u_n$, so that $Y^\perp$ is spanned by $u^1, \ldots, u^\ell$. Then we have the decompositions $V = U \oplus Y$ and $V^* = Y^\perp \oplus U^\perp$ and the identifications $Y^\perp \cong U^*$ and $U^\perp \cong Y^*$.

It is apparent from (1.10) that $\mathrm{B}(\varphi) = \mathrm{B}(\tilde{\varphi})$ if $\varphi - \tilde{\varphi} \in U^\perp$; that is if $\varphi_\varrho = \tilde{\varphi}_\varrho$ for all $\varrho \ge \ell + 1$, so we usually consider $\mathrm{B}(\varphi)$ only for $\varphi \in Y^\perp$.

For our purposes, a "good" basis is one which makes the endomorphisms B_i^λ as structurally similar as possible. For any i, let $\mathbf{W}_i^-$ denote the span of $\{z_1, \ldots, z_{s_i}\}$, Let $\mathbf{W}_i^+$ denote the span of $\{z_{s_i+1}, \ldots, z_r\}$. The map $\mathrm{B}_i^\lambda : W \to W$ has support $\mathbf{W}_\lambda^- \subset W$, and its image lies in $\mathbf{W}_i^+ \subset W$. In order to build this basis from this generic basis, we impose in Section 1(c) additional conditions on the images of the endomorphisms B_i^λ.

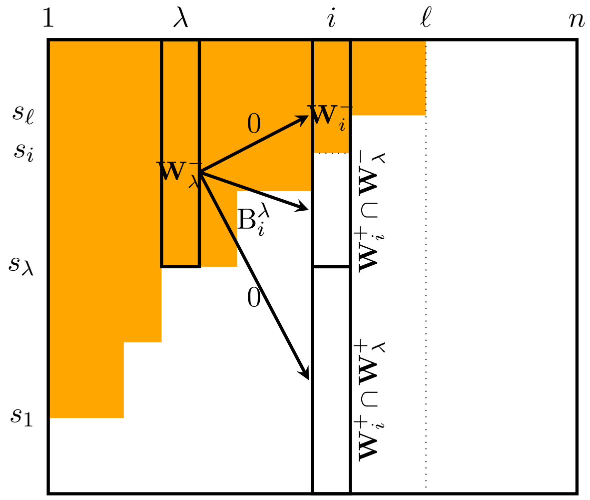

FIGURE 3. The map B_i^λ for an endovolutive tableau.

1(c). Endovolutive Tableaux. A tableau A expressed in bases (u^i) and (z_a) is called *endovolutive*[4] if $B^{a,\lambda}_{i,b} = 0$ for all $a > s_\lambda$. That is, endovolutive means that B_i^λ is an endomorphism of $\mathbf{W}^-_\lambda \subset W$, as in Figure 3.

When considering endovolutive tableaux, it useful to arrange the symbol endomorphisms as an $\ell \times n$ array of $r \times r$ matrices:

$$\begin{bmatrix} I_{s_1} & \mathrm{B}_2^1 & \mathrm{B}_3^1 & \mathrm{B}_4^1 & \cdots & \mathrm{B}_\ell^1 & \cdots & \mathrm{B}_n^1 \\ 0 & I_{s_2} & \mathrm{B}_3^2 & \mathrm{B}_4^2 & \cdots & \mathrm{B}_\ell^2 & \cdots & \mathrm{B}_n^2 \\ 0 & 0 & I_{s_3} & \mathrm{B}_4^3 & \cdots & \mathrm{B}_\ell^3 & \cdots & \mathrm{B}_n^3 \\ 0 & 0 & 0 & I_{s_4} & \cdots & \mathrm{B}_\ell^4 & \cdots & \mathrm{B}_n^4 \\ & & & & \ddots & & \mathrm{B}_i^\lambda & \vdots \\ 0 & 0 & 0 & 0 & 0 & I_{s_\ell} & \cdots & \mathrm{B}_n^\ell \end{bmatrix} \tag{1.11}$$

In (1.11), endovolutivity means that each $r \times r$ sub-matrix in row λ is 0 outside the upper-left $s_\lambda \times s_\lambda$ part. If a tableau is endovolutive in certain bases for W and V^*, then it is also endovolutive under any upper-triangular change-of-basis for $u^i \mapsto g^i_j u^j$. Under such a basis change, the columns of (π^a_i) are linear combinations of the ones to their right, and the sub-matrices in (1.11) change by the corresponding conjugation. (At this point, experts may wish to jump ahead to Theorem 3.16 and Section 3(d).)

Because each B_i^λ is an endomorphism of a particular vector space, it is sensible to consider an eigenvector problem for this map: For any λ, let

$$\mathbf{W}^1(u^\lambda) = \left\{ w \in \mathbf{W}^-_\lambda \ : \ \mathrm{B}^\lambda_\mu w = \delta^\lambda_\mu w \ \forall \mu \leq \ell \right\}. \tag{1.12}$$

That is, we want to find the vectors that are preserved by $\mathrm{B}^\lambda_\lambda$ but are annihilated by B^λ_μ for $\mu \neq \lambda$. More generally, for any $\varphi \in U^*$, let $\mathbf{W}^-(\varphi) =$

[4] The term *endovolutive* was coined in [Smi15], but the phenomenon was described previously in [BCG+90, Chapter IV§5], [Yan87], and it is certainly familiar to anyone who was manipulated tableaux of linear Pfaffian systems.

$\mathbf{W}^-_{\underline{\lambda}}$ where $\underline{\lambda}$ is the minimum index such that $\varphi_{\underline{\lambda}} \neq 0$. Also, let

$$\mathbf{W}^1(\varphi) = \left\{ w \in \mathbf{W}^-(\varphi) \ : \ \left(\sum_\lambda \varphi_\lambda \, \mathrm{B}^\lambda_\mu - \varphi_\mu I \right) w = 0 \ \forall \mu \leq \ell \right\}. \tag{1.13}$$

Equation (1.13) is really saying that $\mathrm{B}(\varphi)(\cdot)w$ is rank-one when restricted to U^*, so we can rewrite it as

$$\mathbf{W}^1(\varphi) = \left\{ w \in \mathbf{W}^-(\varphi) \ : \ z \otimes \varphi + J^a_\varrho (z_a \otimes u^\varrho) \in A_e, \ \exists J \in W \otimes U^\perp \right\}. \tag{1.14}$$

This space is the focus of [Gui68], and it plays an important part in our story. Unlike $\mathbf{W}^-(\varphi)$, its definition does not depend on the basis; its definition depends only on the splitting $V = U \oplus Y$. Its dimension is an important invariant.

LEMMA 1.15. *Suppose that the tableau A admits endovolutive coordinates. For generic φ, $\dim \mathbf{W}^1(\varphi) = s_\ell$.*

Lemma 1.15 is the result of a quick rank computation using (1.13). See details in [Smi15].

Our "good" basis will be built on the requirement that the maps B^λ_i commute on certain combinations of these spaces (and thus share generalized eigenspaces and Jordan-block normal form there). That is, we are aiming for something like the commutative subalgebras seen in [Ger61] and [GS00]. Endovolutivity allows surprisingly direct computation of the desired conditions. For more detail on endovolutivity, see [Smi15] and the references therein. We return to this topic in Section 3(c).

2. Grassmann and Universal Bundles

The Grassmann variety is the set $\mathrm{Gr}_n(X)$ of n-planes in an $(n+r)$-dimensional vector space X. It is a smooth projective variety and a smooth manifold of dimension nr. EDS is a theory of subvarieties of the Grassmann, and this section highlights its most useful features.

2(a). Tangent and Arctangent. The tangent space of $\mathrm{Gr}_n(X)$ is easy to understand in the following way. For any $e \in \mathrm{Gr}_n(X)$, choose a basis[5] $(u_i) = (u_1, \ldots, u_n)$ for e, and choose $(z_a) = (z_1, \ldots, z_r)$ to complete a basis of the entire vector space X. Any n-plane $\tilde{e}$ near e admits a basis $(\tilde{u}_i) = (\tilde{u}_1, \ldots, \tilde{u}_n)$ that we may assume is related to v by a matrix in reduced column echelon form:

$$\begin{pmatrix} \tilde{u}_1 & \ldots & \tilde{u}_n \end{pmatrix} = \begin{pmatrix} u_1 & \ldots & u_n & z_1 & \cdots & z_r \end{pmatrix} \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \\ K^1_1 & \cdots & K^1_n \\ \vdots & \ddots & \vdots \\ K^r_1 & \cdots & K^r_n \end{pmatrix} \tag{2.1}$$

[5]This notation indicates an ordered basis, not a vector. Each u_i is an element of X.

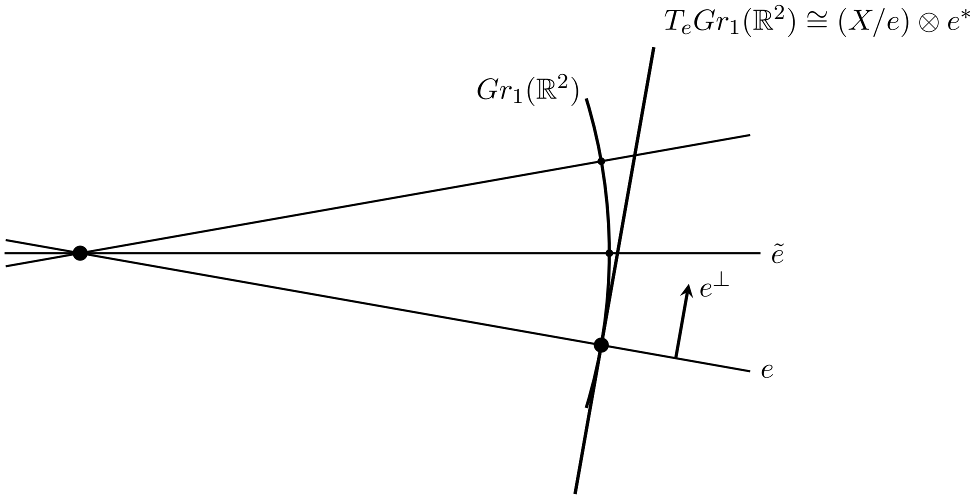

FIGURE 4. From e, identify a nearby line $\tilde{e}$ in $\mathbb{R}^2$ with a relative angle. The map from $T_e\,\mathrm{Gr}_1(\mathbb{R}^2) \cong (-\infty, \infty)$ to the neighborhood $(-\pi/2, \pi/2)$ of e in $Gr_1(\mathbb{R}^2)$ is $\arctan_e$. Its inverse is $\tan_e$.

More succinctly, using the summation convention:

$$\tilde{u}_i = u_i + A_i^a z_a = \delta_i^j u_j + K_i^a z_a. \tag{2.2}$$

That is, $(\tilde{u}_i)$ and (u_i) are related by an $(n+r) \times n$ matrix of rank n whose range $\langle \tilde{u}_1, \dots \tilde{u}_n \rangle = \tilde{e}$ is determined uniquely by the $r \times n$ submatrix (K_i^a). In this sense, $T_e\,\mathrm{Gr}_n(X)$ is isomorphic to the space of $r \times n$ matrices.

However, this isomorphism is not *natural* for an abstract vector space (without metric) because it depends on a choice of splitting $X = e \oplus (X/e)$ by choosing the complementary basis (z_a). To avoid splitting, we can use the dual[6] short-exact sequences

$$\begin{gathered} 0 \to e \to X \to X/e \to 0 \\ 0 \to e^\perp \to X^* \to e^* \to 0 \end{gathered} \tag{2.3}$$

without splitting. Choose any basis $(\theta^a) = (\theta^1, \dots, \theta^r)$ of the annihilator space $e^\perp = (X/e)^*$, and let $(z_a) = (z_1, \dots, z_r)$ be the corresponding dual basis of (X/e). Then, we may take the coefficients K_i^a of

$$K = z_a \otimes K_i^a(\tilde{e}) = z_a \otimes \theta^a(\tilde{u}_i) \in (X/e) \otimes e^* \tag{2.4}$$

as nr local coordinates on $T_e\,\mathrm{Gr}_n(X)$. Moreover, an explicit choice of bases (u_i) for e and (θ^a) for $e^\perp$ is unnecessary. Instead, we need only the abstract homomorphism $K \in (X/e) \otimes e^*$, as the space $\tilde{e} = \langle (\tilde{u}_i) \rangle$ is invariant under $GL(n)$ transformations on (u_i) and $(\tilde{u}_i)$ as well as $GL(r)$ transformations on θ.

[6] Recall that $(X/e)^*$ is canonically isomorphic to $e^\perp$: If $[v] = \{u + e\} \in X/e$, then $\varphi([v]) = \varphi(v) + 0$ is well-defined for all $\varphi \in e^\perp$.

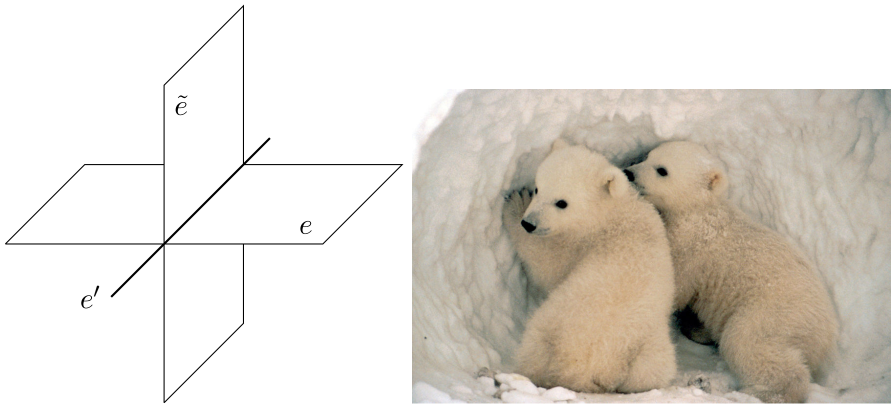

FIGURE 5. Polar pairs.

As in Figure 4, the derivative map $\mathrm{Gr}_n(X) \to (X/e) \otimes e^*$ near e can be seen as a multidimensional generalization of the tangent function, so the inverse map[7] is written $\arctan_e : (X/e) \otimes e^* \to \mathrm{Gr}_n(X)$.

The reader is encouraged to read [MS74, §5] and [KN63] and to search for the terms *Plücker embedding* and *Stiefel manifold* for more detail on this subject.

REMARK 2.5. Notice that any linear subspace of $(X/e) \otimes e^*$ is a tableau in the sense of Section 1. In some sense, it is the *only* example, as arbitrary V and W could be studied by setting $X = V \oplus W$. Moreover, any smooth submanifold $Z \subset \mathrm{Gr}_n(X)$ with tangent space $T_e Z \subset T_e \mathrm{Gr}(X)$ at $e \in Z$ gives $T_e Z$ as a tableau in $(X/e) \otimes e^*$. This observation is the heart of the entire subject.

2(b). Polar pairs. Suppose that $e, \tilde{e} \in \mathrm{Gr}_n(X)$, and that they share a hyperplane. That is, suppose $e' = e \cap \tilde{e}$ and $\dim e' = n - 1$. We call the n-planes e and $\tilde{e}$ *polar pairs* because they are both polar extensions of e'.

Suppose that $\tilde{e}$ is near e in the sense that $\tilde{e} = \arctan_e(K)$ for some K. Let $u_1 \ldots u_{n-1}$ be a basis for e', and extend that basis to a basis $u_1, \ldots, u_{n-1}, u_n$ for e and to a basis $u_1, \ldots, u_{n-1}, \tilde{u}_n$ for $\tilde{e}$. Writing (2.2) in this case, it is apparent that only the nth column of (K^a_i) is nonzero. That is, the tangent homomorphism $K \in (X/e) \otimes e^*$ is rank-1. It cannot be rank-0 unless $e = \tilde{e}$.

Conversely, suppose that $K \in (X/e) \otimes e^*$ is rank-1, and let $\tilde{e} = \arctan_e(K)$. Let $e' = \ker K$, which is a subspace of e of dimension $n - 1$. Any $v \in e'$ is preserved by the map $e \to X$ defined by the matrix in (2.2); hence, $e' \subset \tilde{e}$.

One can see immediately that this generalizes by replacing 1 with any rank k to obtain a Grassmannian version of the rank-nullity theorem

LEMMA 2.6. *If* $e \in \mathrm{Gr}_n(X)$ *and* $\tilde{e} = \arctan_e(K)$ *and* $\operatorname{rank} K = k$, *then*
$$\dim(e \cap \tilde{e}) = n - k.$$

[7] The map $\arctan_e$ is analogous to exponential map $\exp_p : T_p M \to M$ from Riemannian geometry or Lie group representation theory, except that $\arctan_e$ does not make explicit use of a metric or group structure.

It is useful to rephrase Lemma 2.6 to a coordinate-free setting, as Lemma 2.7. This requires dropping the assumption of nearness, so it is possible that K is not unique. This is analogous to failure of injectivity at large distances for the exponential map in Riemannian geometry. Again, the argument is simply a repeated use of the rank-nullity theorem for the short-exact sequences (2.3).

For any $e \in \mathrm{Gr}_n(X)$, let $\mathrm{Pol}_k(e) = \{\tilde{e} \in \mathrm{Gr}_n(X) \ : \ \dim(\tilde{e} \cap e) = n - k\}$. Note that $\mathrm{Pol}_k(e)$ is nonempty if and only if $k \leq r$.

Lemma 2.7. *For any $\tilde{e} \in \mathrm{Pol}_k(e)$, the space $(\tilde{e} \cap e)^{\perp} / e^{\perp} \subset X^*/e^{\perp} = e^*$ has dimension k. The space $\tilde{e}/e \subset X/e$ also has dimension k. This yields the incidence correspondence in Figure 6.*

$$\begin{tikzcd} & \mathrm{Pol}_k(e) \arrow[dl] \arrow[dr] & \\ \mathrm{Gr}_k(X/e) & & \mathrm{Gr}_k(e^*) \end{tikzcd} \qquad \begin{tikzcd} & \tilde{e} \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ \tilde{e}/e & & (\tilde{e} \cap e)^{\perp}/e^{\perp} \end{tikzcd}$$

Figure 6. The incidence correspondence of polar pairs e and $\tilde{e}$.

Now, reconsider the case $k = 1$. Then each $\tilde{e} \in \mathrm{Pol}_1(e)$ yields a hyperplane $e' = \tilde{e} \cap e$. The right image $(e')^{\perp}/e^{\perp}$ in Figure 6 is some line $\xi \in \mathbb{P}e^*$. The left image $\tilde{e}/e$ is some line $w \in \mathbb{P}(X/e)$. So, each $\tilde{e} \in \mathrm{Pol}_1(e)$ yields a rank-one projective homomorphism $w \otimes \xi \in \mathbb{P}\left((X/e) \otimes e^*\right)$. Any element of $\mathbb{P}\left((X/e) \otimes e^*\right)$ could be obtained this way by appropriate choice of $\tilde{e}$.

We can write $w \otimes \xi$ like this: Let $(\omega^1, \ldots \omega^n, \theta^1, \ldots, \theta^r)$ be a basis for X^* such that $e = \ker\{\theta^1, \ldots, \theta^r\}$ and $e' = \ker\{\theta^1, \ldots, \theta^r, \xi\}$ for some $\xi = \xi_i \omega^i$. Then, $\tilde{e} = \ker\{\tilde{\theta}^1, \ldots, \tilde{\theta}^r\}$ for some $\tilde{\theta}^a = J^a_b \theta^b + K^a_i \omega^i$. Because $e' \subset \tilde{e}$, it must be that

$$\begin{aligned} J^a_b \theta^b + K^a_i \omega^i &\equiv 0 \quad \mod \{\theta^c, \xi\}, \text{ so} \\ K^a_i \omega^i &\equiv 0 \quad \mod \{\theta^c, \xi\}, \text{ so} \\ K^a_i \omega^i &\equiv 0 \quad \mod \{\xi\}. \end{aligned} \tag{2.8}$$

Hence, each $K^a_i \omega^i$ is a multiple of ξ; call it $w^a \xi$. Note that $w^a = 0$ for all a if and only if $\tilde{e} = e$, which contradicts our assumption $\dim e' = n - 1$. Choose a basis (z_a) of X/e dual to (θ^a). Let (ω^i) denote the basis of $e^* = X^*/e^{\perp}$ induced by $w^i \in X^*$, so that ξ also denotes the image of $\xi \in X^*$. Let $w = w^a z_a$. Then the induced homomorphism

$$K = z_a \otimes K^a_i \omega^i = z_a \otimes w^a \xi = w \otimes \xi \in (X/e) \otimes e^* \tag{2.9}$$

has rank 1. Each of w and ξ is defined up to scale, so K is well-defined only up to scale, $[K] = \mathbb{P}\left((X/e) \otimes e^*\right)$.

It may be that $\tilde{e}$ lies outside the image of the injective map $\arctan_e$. How then do we interpret K? For some vectors v and $\tilde{v}$, we may write $e = e' + \langle v \rangle$ and $\tilde{e} = e' + \langle \tilde{v} \rangle$ and define a curve $e_\lambda = e' + \langle (1-\lambda)v + \lambda \tilde{v} \rangle$ in $\mathrm{Gr}_n(X)$.

Even if $\tilde{e} = e_1$ lies outside the image of $\arctan_e$, all e_λ lie inside the image of $\arctan_e$ for an open ray of sufficiently small λ. Since K is defined only to scale, the projective homomorphism $w \otimes \xi$ is shared by the all those e_λ. So, the image $\arctan_e(w \otimes \xi)$ contains an open set of $\{e_\lambda\}$ where $e_\lambda \cap e = e'$. In Figure 5, e_λ is the family obtained by rotating e about the axis e' toward $\tilde{e}$.

LEMMA 2.10. *For any* $w \otimes \xi \in \mathbb{P}((X/e) \otimes e^*)$, *there exists a ray of* $K \in T_e \operatorname{Gr}_n(X)$ *representing* $w \otimes \xi$ *such that each* $\tilde{e} = \arctan_e(K)$ *lies in* $\operatorname{Pol}_1(e)$. *That is,* $w \otimes \xi$ *is represented by* $\operatorname{Pol}_1(e)$ *in any open set of* $\operatorname{Gr}_n(X)$ *containing* e. *This also holds for any linear subspace of* $T_e \operatorname{Gr}_n(X)$ *and corresponding submanifold of* $\operatorname{Gr}_n(X)$.

This is sufficient for our purposes, but those seeking a more detailed understanding of polar pairs are encouraged to investigate *Schubert varieties*—for example in [Rob12]—and the other outgrowths of Hilbert's 15th problem.

2(c). The Tautological Bundle. Soon, we will consider algebraic equations defined on e^*. To facilitate this, for any $e \in \operatorname{Gr}_n(X)$, we consider the complexified projective space $\mathbb{X} = \mathbb{P}X \otimes \mathbb{C}$ and its subspace $\mathbb{P}e \otimes \mathbb{C}$. For standard complex projective space, we write $\mathbb{P}^d$ for $\mathbb{CP}^d = \mathbb{P}(\mathbb{C}^{d+1})$. That is, $\mathbb{X} \cong \mathbb{P}^{n+r-1}$.

If we consider all such spaces across all e simultaneously, we obtain the *tautological bundle*[8] $\boldsymbol{\gamma}$ with fiber

$$\boldsymbol{\gamma}_e = \mathbb{P}e \otimes \mathbb{C}, \qquad \forall e \in \operatorname{Gr}_n(X), \tag{2.11}$$

and its dual $\boldsymbol{\gamma}^*$ with fiber

$$\boldsymbol{\gamma}_e^* = \mathbb{P}e^* \otimes \mathbb{C}, \qquad \forall e \in \operatorname{Gr}_n(X), \tag{2.12}$$

and its annihilator $\boldsymbol{\gamma}^\perp$ with fiber

$$\boldsymbol{\gamma}_e^\perp = \mathbb{P}e^\perp \otimes \mathbb{C}, \qquad \forall e \in \operatorname{Gr}_n(X), \tag{2.13}$$

and its cokernel $\mathbb{X}/\boldsymbol{\gamma}$ with fiber

$$(\mathbb{X}/\boldsymbol{\gamma})_e = \mathbb{P}(X/e) \otimes \mathbb{C}, \qquad \forall e \in \operatorname{Gr}_n(X). \tag{2.14}$$

See Figure 7. There is a dual pair of short exact sequences of bundles, analogous to (2.3).

$$\begin{aligned} 0 \to \boldsymbol{\gamma}_e \to \mathbb{X} \to (\mathbb{X}/\boldsymbol{\gamma})_e \to 0, \\ 0 \to \boldsymbol{\gamma}_e^\perp \to \mathbb{X}^* \to \boldsymbol{\gamma}_e^* \to 0. \end{aligned} \tag{2.15}$$

Hence, $\mathbb{P}T_e \operatorname{Gr}(X) \otimes \mathbb{C}$ is isomorphic (naturally) to $(\mathbb{X}/\boldsymbol{\gamma})_e \otimes \boldsymbol{\gamma}_e^*$. If we choose a section of these sequences, then we obtain dual bases to establish an (unnatural) isomorphism $\mathbb{P}X \otimes \mathbb{C} \cong \boldsymbol{\gamma}_e \oplus (\mathbb{X}/\boldsymbol{\gamma})_e$.

[8] These are also called universal bundles or canonical bundles. They are analogous to the sheaves $\mathscr{O}(-1)$ and $\mathscr{O}(1)$, respectively, for varieties in projective space.

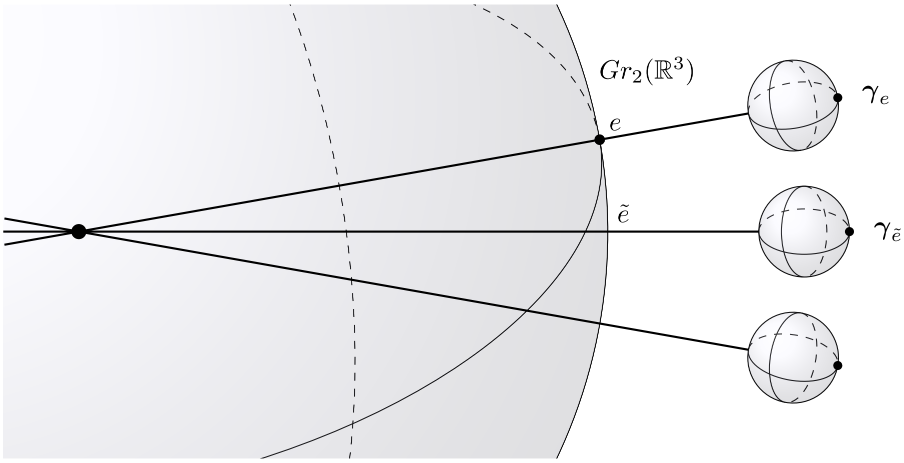

FIGURE 7. A cartoon of the tautological bundle, $\boldsymbol{\gamma}$. Here e is a real 2-plane in $\mathbb{R}^3$, which can be represented by a line because $Gr_2(\mathbb{R}^3) \cong \mathbb{P}(\mathbb{R}^{3*})$. Each $\boldsymbol{\gamma}_e \cong \mathbb{P}(\mathbb{R}^2) \otimes \mathbb{C} = \mathbb{P}(\mathbb{C}^2)$ is a Riemann sphere. Thus, $\boldsymbol{\gamma}$ is depicted as a bundle of 2-spheres over a hemisphere.

One can also consider the frame[9] bundle $\mathcal{F}_{\boldsymbol{\gamma}}$ over $\mathrm{Gr}_n(X)$ associated to $\boldsymbol{\gamma}$, whose fiber is all linear isomorphisms

$$\mathcal{F}_{\boldsymbol{\gamma},e} = \{(u^i) : \boldsymbol{\gamma}_e \xrightarrow{\sim} \mathbb{P}^{n-1}\} = \{\text{bases of } \boldsymbol{\gamma}_e^*\} \cong PGL(n), \tag{2.16}$$

and the coframe bundle $\mathcal{F}_{\boldsymbol{\gamma}^*}$ over $\mathrm{Gr}_n(X)$ associated to $\boldsymbol{\gamma}^*$, whose fiber is all linear isomorphisms

$$\mathcal{F}_{\boldsymbol{\gamma}^*,e} = \{(u_i) : \boldsymbol{\gamma}_e^* \xrightarrow{\sim} \mathbb{P}^{n-1}\} = \{\text{bases of } \boldsymbol{\gamma}_e\} \cong PGL(n). \tag{2.17}$$

To write homogeneous algebraic ideals on $\boldsymbol{\gamma}_e^*$ that vary across $e \in \mathrm{Gr}_n(X)$, the appropriate ring is therefore

$$S = \mathbb{C}[u_1, \ldots, u_n], \text{ for } (u_i) \text{ a section of } \mathcal{F}_{\boldsymbol{\gamma}^*}. \tag{2.18}$$

2(d). Bundles upon Bundles. If M is a smooth manifold of dimension $m = n + r$, then we can form the smooth bundle $\mathrm{Gr}_n(TM)$ with fiber $\mathrm{Gr}_n(T_pM)$. Let $\varpi : \mathrm{Gr}_n(TM) \to M$ denote the bundle projection.

Because (2.3) holds across the bundle for $X = T_pM$, any local section of $\mathrm{Gr}_n(TM)$ can be described by choosing its annihilator section of $\mathrm{Gr}_r(T^*M)$, and vice-versa.

For every $p \in M$, its Grassmann fiber $\mathrm{Gr}_n(T_pM)$ has a tautological bundle $\boldsymbol{\gamma}(p)$ with fiber $\boldsymbol{\gamma}_e(p) = \mathbb{P}e \otimes \mathbb{C}$. The total space $\mathrm{Gr}_n(TM)$ is a manifold in its own right. Hence, we may consider $\boldsymbol{\gamma}$ as a bundle over the manifold $\mathrm{Gr}_n(TM)$, which is itself a bundle over M. In other words, we can reinterpret Section 2(c) where $\mathbb{X}_e = \mathbb{P}T_pM \otimes \mathbb{C}$ is the fiber of the bundle $\mathbb{X}$ over $\mathrm{Gr}_n(TM)$ at e with $\varpi(e) = p$. A complete description of some $v \in \boldsymbol{\gamma}$

[9]Some authors might flip the names of the frame and coframe bundles. I tend to choose this notation because the frame bundle is covariant with diffeomorphisms on the base space, and only contravariant objects get a "co-" prefix. The jargon for duality is always frustrating.

$$\begin{tikzcd}
\boldsymbol{\gamma} \arrow[r, dash, "\supset"] \arrow[d] & \boldsymbol{\gamma}(p) \arrow[r, dash, "\supset"] \arrow[d] & \mathbb{P}e \otimes \mathbb{C} \arrow[r, dash, "\ni"] \arrow[d] & v \arrow[dl] \\
\mathrm{Gr}_n(TM) \arrow[r, dash, "\supset"] \arrow[d, "\varpi"] & \mathrm{Gr}_n(T_pM) \arrow[r, dash, "\ni"] \arrow[d, "\varpi"] & e \arrow[dl, "\varpi"] & \\
M \arrow[r, dash, "\ni"] & p & &
\end{tikzcd}$$

FIGURE 8. Tautological bundles over Grassmann bundles over manifolds. Vertical arrows are bundle projections.

would be (p, e, v) where $u \in \mathbb{P}e \otimes \mathbb{C}$, and $e \in \mathrm{Gr}_n(T_pM)$, and $p \in M$. See Figure 8. Analogous constructions hold for $\boldsymbol{\gamma}^*$, $\boldsymbol{\gamma}^\perp$, $(\mathbb{X}/\boldsymbol{\gamma})$, $\mathcal{F}_{\boldsymbol{\gamma}}$, and $\mathcal{F}_{\boldsymbol{\gamma}^*}$ from Section 2(c). To write homogeneous algebraic ideals on $\boldsymbol{\gamma}_e^*$ that vary across $e \in \mathrm{Gr}_n(TM)$, the appropriate ring is therefore

$$S = C^\infty(M)[u_1, \ldots, u_n] \otimes \mathbb{C}, \text{ for } (u_i) \text{ a section of } \mathcal{F}_{\boldsymbol{\gamma}^*}. \tag{2.19}$$

2(e). The Contact Ideal. For any $e \in \mathrm{Gr}_n(TM)$, consider its annihilator subspace $e^\perp \subset T_p^*M$. There is a corresponding subspace $J_e \subset T_e^* \mathrm{Gr}_n(TM)$, defined as

$$J_e = \left\langle \zeta \circ \varpi_* \text{ such that } \zeta \in e^\perp \right\rangle \tag{2.20}$$

as in Figure 9. If (z^a) is a basis of $e^\perp$, then we can define a basis of J_e by $\theta^a = z^a \circ \varpi_*$.

$$\begin{tikzcd}
e \in \mathrm{Gr}_n(TM) \arrow[d, "\varpi"'] & T_e \mathrm{Gr}_n(TM) \arrow[d, "\varpi_*"'] \arrow[dr, dashed, "\langle(\theta^a)\rangle = J_e"] & \\
p \in M & T_pM \arrow[r, "\langle(z^a)\rangle = e^\perp"'] & \mathbb{R}^r
\end{tikzcd}$$

FIGURE 9. Contact forms on the Grassmann bundle of M.

The differential ideal $\mathcal{J} \subset \Omega^\bullet(\mathrm{Gr}_n(TM))$ generated by $\langle \theta^a, \mathrm{d}\theta^a \rangle$ from J is called the *contact ideal*.

Note that, for any (local) section $\epsilon : M \to \mathrm{Gr}_n(TM)$, the contact ideal satisfies the universal reproducing property $\epsilon^*(J) = \epsilon^\perp$. However, even if the topology of M forces the section ϵ to be defined locally, the module J is defined globally across $\mathrm{Gr}_n(TM)$.

$$\begin{tikzcd} & \boldsymbol{\gamma}^* \arrow[d] \\ & \mathrm{Gr}_n(TM) \arrow[d] \\ N \arrow[ur, dashed, "\iota^{(1)}"] \arrow[r, "\iota"] & M \end{tikzcd}$$

FIGURE 10. Dual tautological bundle pulls back to an immersed submanifold.

If one were to choose local coordinates (x^i, y^a) for M and local fiber coordinates (P^a_i) for $\mathrm{Gr}_n(T_{(x,y)}M)$ near a particular n-plane $e = \ker\{\mathrm{d}y^a\}$, then $\mathcal{J}$ is the differential ideal typically written as

$$\begin{cases} 0 = \theta^a = \mathrm{d}y^a - P^a_i \mathrm{d}x^i, \\ 0 = \mathrm{d}\theta^a = \mathrm{d}P^a_i \wedge \mathrm{d}x^i, \end{cases} \tag{2.21}$$

where the functions P^a_i depend on $e \in \mathrm{Gr}_n(TM)$.

After reading Section 2(f), compare this coordinate description to your favorite definition of jet space, $\mathbb{J}^1(\mathbb{R}^n, \mathbb{R}^r)$. Also, compare the local fiber coordinates P^a_i to the tangent coordinates K^a_i from Section 2(a). For some highly amusing applications of the contact system, see [Gro86].

2(f). Immersions and Frame Bundles. Suppose that $\iota : N \to M$ is an immersion, and that $\dim N = n$. For any $x \in N$ with $\iota(x) = p$, the push-forward derivative has image $\iota_*(T_xN)$, which is an n-dimensional subspace of T_pM; hence, $\iota_*(T_xN) \in \mathrm{Gr}_n(TM)$. Define the map $\iota^{(1)} : N \to \mathrm{Gr}_n(TM)$ by $\iota_*(T_xN) \in \mathrm{Gr}_n(TM)$. Define the map $\iota^{(1)} : N \to \mathrm{Gr}_n(TM)$ by

$$\iota^{(1)}(y) = \iota_*(T_yN) = e \in \mathrm{Gr}_n(TM), \tag{2.22}$$

and note that $\iota = \varpi \circ \iota^{(1)}$.

It is obvious from the definition that $\iota^{(1)}$ is also an immersion. Therefore, we can use it to pull-back the tautological bundle $\boldsymbol{\gamma}^*$ as defined in Sections 2(c) and 2(d). Let $\boldsymbol{\gamma}^*_N = \iota^{(1)*}\boldsymbol{\gamma}^*$, which has fiber

$$\boldsymbol{\gamma}^*_{N,y} = \boldsymbol{\gamma}^*_e(p) = \mathbb{P}e^* \otimes \mathbb{C} = \mathbb{P}\iota_*(T_yN) \otimes \mathbb{C}; \tag{2.23}$$

that is, $\boldsymbol{\gamma}^*_N$ is identified with $\mathbb{P}T^*N \otimes \mathbb{C}$ via ι_*. See Figure 10.

The immersion $\iota^{(1)}$ is called the *prolongation* of the immersion ι.

Now, consider the contact forms (θ^a) forms from Section 2(e). For all $x \in N$ and all $v \in T_xN$, we have

$$\iota^{(1)*}(\theta^a)(v) = \theta^a(\iota^{(1)}_*(v)) = z^a \circ \varpi_* \circ \iota^{(1)}_*(v) = z^a(\iota_*(v)) = 0, \tag{2.24}$$

which ultimately gives the following lemma:

Lemma 2.25. *If $\iota : N \to M$ is an immersion for $\dim N = n$, then $\iota^{(1)*}(\mathcal{J}) = 0$. Conversely, if $\iota' : N \to M^{(1)}$ is an immersion for $\dim N = n$ satisfying $\iota'^*(\mathcal{J}) = 0$ and such that the image $\iota'_*(T_xN)$ is transverse to the fiber $\ker \varpi_*$ for all $x \in N$, then there is some immersion $\iota : N \to M$ such that $\iota^{(1)} = \iota'$.*

$$\begin{tikzcd} u \in \mathcal{F}N \arrow[dd, "\varpi"'] \\ \\ x \in N \end{tikzcd} \qquad \begin{tikzcd} T_u\mathcal{F}N \arrow[dd, "\varpi_*"'] \arrow[dr, dashed, "(\omega^i)"] & \\ & \mathbb{P}^{n-1} \\ T_xN \arrow[ur, "(u^i)"'] & \end{tikzcd}$$

Figure 11. Tautological Form of the frame bundle of a manifold N.

Moreover, recall that any manifold N of dimension n admits a projective frame bundle $\varpi : \mathcal{F}N \to N$ with fiber

$$\mathcal{F}_xN = \{(u^i) : T_xN \xrightarrow{\sim} \mathbb{P}^{n-1}\} = \{\text{bases of } T_x^*N\}. \cong PGL(n), \tag{2.26}$$

The total space $\mathcal{F}N$ admits a tautological[10] 1-form $\omega : T_u\mathcal{F}N \to \mathbb{P}^{n-1}$ defined by $\omega^i_u = u^i \circ \varpi_*$ as in Figure 11. It is characterized by its universal reproducing property $\eta^*(\omega) = \eta$ for any (local) section $\eta : N \to \mathcal{F}N$. However, even if the topology of N forces the 1-form η to be defined locally, the 1-form ω is defined globally on $\mathcal{F}N$.

For any diffeomorphism $f : N \to \tilde{N}$, there is an induced (covariant) map on the frame bundles $f^\dagger : \mathcal{F}N \to \mathcal{F}\tilde{N}$ by $f^\dagger : (u^i) \mapsto (u^i) \circ (f_*)^{-1}$. Using the universal property, it is easy to prove this lemma, which shows that diffeomorphisms are characterized by the tautological form on the frame bundle:

Lemma 2.27. *If $f : N \to \tilde{N}$ is a diffeomorphism, then $(f^\dagger)^*(\tilde{(\omega)}) = \omega$. Conversely, if $F : \mathcal{F}N \to \mathcal{F}\tilde{N}$ is $PGL(n)$-invariant diffeomorphism such that $F^*(\tilde{\omega}) = \omega$, then there exists a unique diffeomorphism $f : N \to \tilde{N}$ such that $f^\dagger = F$.*

Combining the universal properties of the $\mathcal{J}$ and ω, we obtain the following theorem telling us what information we can transfer from $\mathrm{Gr}_n(TM)$ to an immersed submanifold:

Theorem 2.28. *If $\iota : N \to M$ is a smooth immersion, then*

- *$\iota^{(1)*}(\mathcal{J}) = 0$, and*
- *$\mathcal{F}N = \iota^{(1)*}(\mathcal{F}_\gamma)$.*

Conversely, if $\iota' : N \to \mathrm{Gr}_n(TM)$ is a smooth immersion such that

[10] In various references, this 1-form is called the canonical 1-form, the Hilbert 1-form, and the soldering 1-form.

- $\iota'^*(\mathcal{J}) = 0$, *and*
- $\mathcal{F}N = \iota'^*(\mathcal{F}_\gamma)$,

then there is some smooth immersion $\iota : N \to M$ *such that* $\iota^{(1)} = \iota'$.

That is, an immersed submanifold satisfies the contact ideal, which is generated differentially by some annihilator 1-forms (θ^a) spanning $\boldsymbol{\gamma}^\perp$, and its frame bundle is equipped with tautological 1-forms (ω^i) spanning $\boldsymbol{\gamma}^*$.

Remark 2.29. Note the similarity between the universal property of the contact ideal on the Grassman bundle and the universal property of the tautological 1-form on the frame bundle. Exploitation of this interaction as in Theorem 2.28 has a long and interesting history.

For example, consider the study of a Lie pseudogroup acting on a manifold M. One option is to differentiate the coordinates of M repeatedly until differential syzygies of the Lie pseudogroup action can be found in prolonged local coordinates, which are then converted to a coordinate-free description using the pseudogroup action. The other option is to work on the frame bundle of M immediately, which is automatically invariant, then prolong as necessary to reveal the syzygies. The latter is used often when the Lie pseudogroup arises as equivalence of intrinsic G-structures, and the former is used often when the Lie pseudogroup arises from an extrinsic action on some ambient coordinates. For more on these fascinating and interconnected ideas, I encourage you to read [**Cle**], [**Olv95**], [**Val13**], and [**Gar89**]—and the collected works of E. Cartan.

3. Exterior Differential Systems

Let M be a smooth manifold of finite dimension m. An *exterior differential system* [EDS] on M consists of an ideal $\mathcal{I}$ in the total exterior algebra $\Omega^\bullet(M)$ that is differentially closed and finitely generated. *Differentially closed* means that $\mathrm{d}\mathcal{I} \subset \mathcal{I}$. *Finitely generated* means that in each degree p, the p-forms in the ideal, $\mathcal{I}_p = \mathcal{I} \cap \Omega^p(M)$, form a finitely generated $C^\infty(M)$-module. We assume that $\mathcal{I}_0 = 0$; otherwise, one would restrict to a submanifold defined by those functions. Optionally, we sometimes specify an independence condition as an n-form $\boldsymbol{\omega} \in \Omega^n(M)$ that is not allowed to vanish on solutions.

3(a). Integral Elements and Integral Manifolds. Why would anyone define such an object? In Section 2, we explored the geometry of the bundle $\mathrm{Gr}_n(TM)$. Exterior differential systems provide a convenient language to study the geometry of *smooth sub-bundles* $M^{(1)}$ of $\mathrm{Gr}_n(TM)$.

To be precise, an *integral element* of $\mathcal{I}$ at $p \in M$ is a linear subspace $e \subset T_pM$ such that $\varphi|_e = 0$ for all $\varphi \in \mathcal{I}_n$. That is, the n-forms in $\mathcal{I}$ provide a collection of functions that cut out a variety, $\mathrm{Var}_n(\mathcal{I}) \subset \mathrm{Gr}_n(TM)$. These functions vary smoothly in M and are homogeneous in the fiber variables.

There is a maximal dimension n for which $\mathrm{Var}_n(\mathcal{I})$ is locally non-empty, which is the case of interest. If an independence condition $\boldsymbol{\omega}$ is specified, we also require $\boldsymbol{\omega}|_e \neq 0$, which forces $\mathrm{Var}_n(\mathcal{I})$ to lie in the open subset of $\mathrm{Gr}_n(TM)$ for which that condition holds.

Because $\mathcal{I}_n$ is finitely generated by smooth functions, there is an open, dense subset $\mathrm{Var}_n^o(\mathcal{I}) \subset \mathrm{Var}_n(\mathcal{I})$ defined as the smooth subbundle of $\mathrm{Gr}_n(TM)$ that is cut out smoothly by smooth functions. These are the *Kähler-ordinary* elements. A single connected component of $\mathrm{Var}_n^o(\mathcal{I})$ is called $M^{(1)}$, and we allow ourselves to redefine M so that $\varpi : M^{(1)} \to M$ is a smooth bundle.

Let s denote the dimension of each fiber of the projection $M^{(1)} \to M$, so $t = nr - s$ is the corresponding codimension of $T_e M_p^{(1)}$ in $T_e \mathrm{Gr}_n(T_p M)$. That is, $A_e = \ker \varpi_* = T_e M_p^{(1)}$ is a tableau, and because $M^{(1)}$ is a smooth manifold, we have:

LEMMA 3.1. *$K \in A_e$ implies $\mathrm{arctan}_e(K) \in M^{(1)}$ near e.*

That is, we have a well-defined vector bundle $A = \ker \varpi_* \subset TM^{(1)}$ over $M^{(1)}$. So that we may apply the results of Section 1, we also restrict ourselves to an open subset of $M^{(1)}$ where the Cartan characters of A_e are constant for $e \in M^{(1)}$. This is the assumption that a *regular flag* of e may be chosen smoothly across $e \in M^{(1)}$.

Moreover, define the tautological bundles

$$\begin{aligned} V &= \boldsymbol{\gamma}|_{M^{(1)}} = \{\mathbb{P}e \otimes \mathbb{C}\}, \\ V^* &= \boldsymbol{\gamma}^*|_{M^{(1)}} = \{\mathbb{P}e^* \otimes \mathbb{C}\}, \\ W &= (\mathbb{X}/\boldsymbol{\gamma})|_{M^{(1)}} = \{\mathbb{P}(T_p M/e) \otimes \mathbb{C}\}, \\ V^\perp &= \boldsymbol{\gamma}^\perp|_{M^{(1)}} = \{\mathbb{P}e^\perp \otimes \mathbb{C}\} \end{aligned} \tag{3.2}$$

Sometimes, it is convenient to think of A as being a complex projective bundle, in which case we consider it to be a subbundle of $W \otimes V^*$.

An *integral manifold* of $\mathcal{I}$ is an immersion $\iota : N \to M$ such that $\iota^*(\varphi) = 0$ for all $\varphi \in \mathcal{I}$. (If an independence condition $\boldsymbol{\omega}$ is specified, we require that $\iota^*(\boldsymbol{\omega}) \neq 0$, too.) When we are considering a particular $M^{(1)} \subset \mathrm{Var}_n(\mathcal{I})$ as above, we say N is an *ordinary* integral manifold provided that $\iota_*(TM) \subset M^{(1)}$. All of the observations from Section 2(f) apply, but $\iota^{(1)}(N)$ lies in the submanifold $M^{(1)}$, and $\iota_*^{(1)}(TN)$ lies in the subbundle A. The overall goal is to construct all ordinary integral manifolds of $(M, \mathcal{I})$ through the careful study of the prolongation $M^{(1)}$.

One reason to define exterior differential systems this way is that the term *PDE* or *system of PDEs* is difficult to pin down. Colloquially, "system of PDEs" usually means a finite set of (hopefully, smooth) equations on some jet space. Because the contact system $\mathcal{J}$ on $\mathrm{Gr}_n(TM)$ implies the notion of jet space, a system of PDEs can be represented as an EDS that is generated by the contact system along with a finite set of equations defined locally on $\mathrm{Gr}_n(TM)$.

Even by this definition, an exterior differential system could be rather wild; however, in many practical applications, it happens that $\mathcal{I}$ is generated by a finite collection of differential forms of various degrees, so the fiber $M^{(1)}$ is a smooth algebraic variety in local fiber coordinates near a solution $e \in \mathrm{Gr}_n(TM)$.

3(b). Prolongation and Spencer Cohomology. Suppose that $\iota : N \to M$ is an ordinary integral manifold of $\mathcal{I}$. By Theorem 2.28, the 1-forms θ^a spanning J_e must vanish for each $e \in \iota^{(1)}(N)$. The tautological form (ω^i) on $\mathcal{F}_\mathcal{T}$ pulls back to a nondegenerate frame (η^i) on N, since $\iota^{(1)}$ is an immersion.

Therefore, if $\iota^{(1)} : N \to M^{(1)}$ actually exists, we have

$$\begin{aligned} \iota^{(1)*}(\theta^a) &= 0, \\ \iota^{(1)*}(\mathrm{d}\theta^a) &= 0, \\ \iota^{(1)*}(\omega^1 \wedge \cdots \wedge \omega^n) &= \eta^1 \wedge \cdots \wedge \eta^n \neq 0. \end{aligned} \tag{3.3}$$

However, working on the frame bundle of $M^{(1)}$, these forms satisfy a more general equation

$$\mathrm{d}\theta^a \equiv \pi^a_i \wedge \omega^i + \frac{1}{2} T^a_{i,j}\, \omega^i \wedge \omega^j, \quad \mod \{\theta^b\}. \tag{3.4}$$

For any choice of dual coframe $w_a \leftrightarrow \theta^a$ for $W \leftrightarrow V^\perp$, we can see that $\pi = \pi^a_i (w_a \otimes \omega^i)$ lies in A. In particular, it must be that $\iota^{(1)*}(\pi^a_i) = P^a_{i,j}\eta^j$ for some function $P^a_{i,j}$ that must satisfy $P^a_{i,j}\eta^i \wedge \eta^j = 0$, so $P^a_{i,j} = P^a_{j,i}$. That is, the homomorphism $P \in A \otimes V^*$ lies in the fiber over e of the subbundle

$$A \otimes V^* \subset (W \otimes V^*) \otimes V^* = W \otimes (V^* \otimes V^*). \tag{3.5}$$

Moreover, the existence of an immersion $\iota^{(1)} : N \to M^{(1)}$ requires that the *torsion* term $w_a T^a_{i,j}\, \omega^i \wedge \omega^j$ is zero; that is, it must be possible to rewrite $\pi^a_i = \pi^a_i + Q^a_{i,j}\omega^j$ for $Q \in A \otimes V^*$ such that any $T^a_{i,j}$ term is absorbed. Note that this absorption of torsion is an algebraic property of the tableau A.

In summary,

Lemma 3.6. *Let $\delta : A \otimes V^* \to W \otimes \wedge^2 V^*$ denote the composition of skewing $\otimes^2 V^* \to \wedge^2 V^*$ and inclusion $A \to W \otimes V^*$, and write $A^{(1)} = \ker \delta$ and $H^2(A) = \operatorname{coker} \delta$:*

$$0 \to A^{(1)} \to A \otimes V^* \xrightarrow{\delta} W \otimes \wedge^2 V^* \to H^2(A) \to 0. \tag{3.7}$$

For any ordinary integral manifold N, the bundle (3.5) lies in $A^{(1)}$, and the pullback of torsion T is zero in $H^2(A)$.

Writing δ in a chosen coframe, it is easy to check that

$$\dim A^{(1)} \leq s_1 + 2s_2 + \cdots + ns_n. \tag{3.8}$$

The case of equality is considered in Section 3(c).

The exterior differential system $\mathcal{I}^{(1)}$ on $M^{(1)}$ generated as

$$\mathcal{I}^{(1)} = \langle \theta^a, \mathrm{d}\theta^a \rangle = \varpi^*(\mathcal{I}) + \mathcal{J} \tag{3.9}$$

is called the (first) *prolongation* of $(M, \mathcal{I})$, and we are back where we started in Section 3. We can construct $M^{(2)} \subset \mathrm{Gr}_n(TM^{(1)})$, and repeat the entire process for $E \in M^{(2)}$ over $e \in M^{(1)}$ that was used for $e \in M^{(1)}$ over $p \in M$. Lemma 3.6 essentially says that $A^{(1)}$ is the tableau $T_E M^{(2)}$. Thus, we can construct $M^{(3)}$ over $M^{(2)}$ and re-apply Lemma 3.6 in that case. By the definition of $M^{(1)}$ and (3.9), we have

COROLLARY 3.10. *Every ordinary integral manifold N of $(M^{(1)}, \mathcal{I}^{(1)})$ is also an ordinary integral manifold of $(M, \mathcal{I})$. However, the converse might fail, as the smooth connected locus of $M^{(1)}$ may be a strict subset of* $\mathrm{Var}_n(\mathcal{I})$.

Overall, we achieve exact sequences that summarize the entire situation of the tangent spaces of an immersed ordinary integral manifold N of $\mathcal{I}$, $\mathcal{I}^{(1)}$, $\mathcal{I}^{(2)}$, ...

$$\begin{aligned}
0 \to A \to W \otimes \wedge^1 V^* \to H^1(A) \to 0,\\
0 \to A^{(1)} \to A \otimes V^* \xrightarrow{\delta} W \otimes \wedge^2 V^* \to H^2(A) \to 0,\\
0 \to A^{(2)} \to A^{(1)} \otimes V^* \xrightarrow{\delta} W \otimes \wedge^3 V^* \to H^3(A) \to 0,\\
\vdots\\
0 \to A^{(n-1)} \to A^{(n-2)} \otimes V^* \xrightarrow{\delta} W \otimes \wedge^n V^* \to H^n(A) \to 0.
\end{aligned} \tag{3.11}$$

The cokernels $H^1(A)$, $H^2(A)$, ..., $H^n(A)$ are the *Spencer cohomology* of the tableau A. Even outside the context of exterior differential systems, they are defined for formal tableaux $A \subset W \otimes V^*$ via the exact sequences (3.11) as

$$H^p(A) = \left(A \otimes (\otimes^{p-1} V^*)\right) / \left(W \otimes \wedge^p V^*\right). \tag{3.12}$$

Spencer cohomology finds functional obstructions to the solution of the initial-value problem on $M^{(p)}$ in the form of *torsion*; this is explained nicely in [IL03, Section 5.6].

Spencer cohomology was a major focus of the formal study of partial differential equations and Lie pseudogroups in the mid-20th century; most notably, [Spe62, Qui64, SS65, GQS66, Gol67, Gar67, Gui68, GK68, GQS70]. As it happens, many of the major results of that era are easy to re-prove under our regularity assumptions on $M^{(1)}$ and using the perspective from Section 1, particularly when using the involutivity criteria in Section 3(c) that were detailed in [Smi15]. We demonstrate this in Parts II and III.

3(c). Involutivity of Differential Ideals.

DEFINITION 3.13 (Cartan's test). A tableau A is called *involutive* if equality holds in Equation (3.8).

DEFINITION 3.14. A tableau A is called *formally integrable* if $H^p(A) = 0$ for all $p \geq 2$.

Cartan's test comes from the following consequence of the Cartan–Kähler theorem[11]

THEOREM 3.15. *Suppose that $(M, \mathcal{I})$ is an analytic exterior differential differential system, that $M^{(1)}$ is a smooth sub-bundle, and that the tableau bundle A has constant Cartan characters over $M^{(1)}$. If A is involutive and formally integrable, then for every $x \in M$, there is an analytic ordinary*

[11] See [BCG+90, Chapter III] or [IL03] for more background on the Cartan–Kähler theorem; it is not our focus here.

integral manifold $\iota : N \to M$ through x. Moreover, such N are parametrized locally by r constants, s_1 functions of 1 variable, s_2 functions of 2 variables, $\ldots$, s_ℓ functions of ℓ variables.

Somewhat confusingly, the situation in Theorem 3.15 is called *involutivity of $\mathcal{I}$*; that is, an EDS might fail to be involutive even if its tableau is involutive, because there may be nonzero *torsion* in $H^p(A)$.

For a beautiful interpretation of Cartan's test that is relevant to the later Sections of this course, read the introduction of [Yan87]. In summary, ordinary integral manifolds are constructed using by decomposing the Cauchy problem into a sequence of steps, each of which is determined and has solutions using the Cauchy–Kowalevski theorem.

For fixed spaces W and V^*, involutivity is a closed algebraic condition on tableaux in $W \otimes V^*$. Because the conditions come from Cartan's test, which involves $W \otimes \wedge^2 V^*$, it is not surprising that the conditions are quadratic; however, writing down the precise ideal is a lengthy argument. Doing so was suggested in [BCG+90, Chapter IV§5] and accomplished for general tableaux in [Smi15] following the outline in [Yan87].

Theorem 3.16 (Involutivity Criteria). *Suppose a tableaux is given in a generic basis of V^* by (1.10). The tableaux is* involutive *if and only if there exists a basis of W such that*

(i) B^λ_i *is endovolutive in that basis, and*
(ii) $\left(\mathrm{B}^\lambda_l \mathrm{B}^\mu_k - \mathrm{B}^\lambda_k \mathrm{B}^\mu_l\right)^a_b$ *for all $\lambda < l < k$ and $\lambda \leq \mu < k$ and all $a > s_l$.*

This theorem is our main computational tool in Part II.

3(d). Moduli of Involutive Tableaux. While it seems like a trivial (if lengthy) computation, consider carefully the meaning of Theorem 3.16: We can fix r, n, and Cartan characters $s_1, \ldots, s_n$ and then write down an explicit ideal in coordinates whose variety is all of the involutive tableaux with those characters. Hence, we can use computer algebra systems such as Macaulay2, Magma, and Sage to decompose and analyze that ideal using Gröbner basis techniques. With enough computer memory, we can answer the question "What is the moduli of involutive tableaux?" By virtue of Theorem 3.15, this is very close to answering the question "What is the moduli of involutive PDEs?"

For example, fix $r = n = 3$ and $(s_1, s_2, s_3) = (3, 2, 0)$. An endovolutive tableau must be of the form

(3.17)
$$(\pi^a_i) = \begin{pmatrix} a_0 & a_3 & x_3a_0 + x_6a_1 + x_9a_2 + x_{12}a_3 + x_{14}a_4 \\ a_1 & a_4 & x_4a_0 + x_7a_1 + x_{10}a_2 + x_{13}a_3 + x_{15}a_4 \\ a_2 & x_0a_0 + x_1a_1 + x_2a_2 & x_5a_0 + x_8a_1 + x_{11}a_2 \end{pmatrix},$$

or in block form like (1.11),

$$(\mathrm{B}^\lambda_i) = \begin{bmatrix} \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\x_0&x_1&x_2\end{pmatrix} & \begin{pmatrix}x_3&x_6&x_9\\x_4&x_7&x_{10}\\x_5&x_8&x_{11}\end{pmatrix}\\ \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} & \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} & \begin{pmatrix}x_{12}&x_{14}&0\\x_{13}&x_{15}&0\\0&0&0\end{pmatrix}\end{bmatrix} \tag{3.18}$$

Involutivity is an affine quadratic ideal $\mathscr{G}$ on $\mathbb{C}(x_0,\ldots,x_{15})$ generated by all the terms of $\mathrm{B}^1_2\,\mathrm{B}^2_3 - \mathrm{B}^1_3\,\mathrm{B}^2_2$ as

$$\mathscr{G} = \begin{cases} x_0x_3 + x_1x_4 + x_2x_5 - x_0x_{11},\\ x_0x_6 + x_1x_7 + x_2x_8 - x_1x_{11},\\ x_0x_9 + x_1x_{10},\\ x_0x_{12} + x_1x_{13} - x_5,\\ x_0x_{14} + x_1x_{15} - x_8. \end{cases} \tag{3.19}$$

The complete primary decomposition of this ideal shows two components. The maximal component has dimension 12, and it is described by the fairly boring prime ideal $\{x_0, x_1, x_5, x_8\}$. The other component has dimension 11 and its prime ideal is generated by 27 polynomials. See `http://goo.gl/jGTnMU` for how to compute this in SageMathCell. To break it down further for intuition, suppose that B^2_3 is in Jordan form, so that $x_{13} = 0$ and either ($x_{14} = 1$ and $x_{12} = x_{15}$) or $x_{14} = 0$.

Many of your favorite involutive second-order scalar PDEs in three independent variables live somewhere in this variety; see Sections 4(c). Up to some notion of equivalence, this is essentially the moduli space of such equations. As seen in Part II, their characteristic varieties are obtained by combining $\mathscr{G}$ with the rank-one ideal $\mathscr{R}$ on $\mathbb{C}(x_0,\ldots,x_{15},a_0,\ldots,a_4)$.

However, there is still some ambiguity to be resolved, as it may be that a given abstract tableau admits several endovolutive bases with apparently distinct coordinate descriptions.

3(e). Cauchy retractions. Before proceeding to Part II, it is worthwhile to mention Cauchy retractions, which are much simpler than—and quite distinct from—elements of the characteristic variety. To confuse matters, many references call these "Cauchy characteristics." For any differentially closed ideal $\mathcal{I} \subset \Omega^\bullet M$, the Cauchy retractions are the vectors that preserve $\mathcal{I}$; that is, $\mathfrak{g} = \{v \in TM : v \lrcorner \mathcal{I} \subset \mathcal{I}\}$ Because $\mathcal{I}$ is differentially closed, the annihilator bundle $\mathfrak{g}^\perp \subset T^*M$ is the smallest Frobenius ideal in $\Omega^\bullet(M)$ that contains $\mathcal{I}$. Then, for any integral manifold $\iota : N \to M$, the subspaces $\mathfrak{g} \cap \iota^{(1)}(N)$ form an integrable distribution; that is, $\mathfrak{g}^\perp_N$ is Frobenius as well [Gar67].

Because $\mathfrak{g}^\perp$ is a Frobenius system—a system of ODEs—it is common to redefine $(M, \mathcal{I})$ so that it is free of Cauchy retractions before proceeding to study its integral manifolds. The separation between $\mathfrak{g}^\perp$ and the characteristic variety Ξ is explored further in [Smi14].

Part II

Characteristic and Rank-One Varieties

Thank you for taking the time to read the enormous amount of background in Part I.

Here we stand: We have an exterior differential system $\mathcal{I}$ on M with independence condition ω; perhaps this EDS arose from a system of PDEs on M. That EDS yields a smooth subbundle $M^{(1)} \subset \mathrm{Gr}_n(TM)$, where any $e \in M^{(1)}$ is an integral element of the original EDS. As a manifold in its own right, $M^{(1)}$ is equipped with tautological bundles V, V^*, W, and A with fibers

$$\begin{aligned} V_e &= \mathbb{P}e \otimes \mathbb{C}, \\ V^*_e &= \mathbb{P}e^* \otimes \mathbb{C}, \\ W_e &= \mathbb{P}(T_pM/e) \otimes \mathbb{C}, \text{ and} \\ A_e &= \mathbb{P}T_eM^{(1)}_p \otimes \mathbb{C}, \end{aligned} \tag{3.20}$$

respectively. Moreover, A is a subbundle of $W \otimes V^*$, so it is a tableau bundle. Its symbol σ gives a short-exact sequence of bundles:

$$0 \to A \to W \otimes V^* \xrightarrow{\sigma} H^1(A) \to 0. \tag{3.21}$$

An integral manifold is an immersion $\iota : N \to M$ such that $\iota_*(T_pN) \in M^{(1)}_{i(p)}$ for all $p \in N$. Let $\iota^{(1)} : N \to M^{(1)}$ denote the map $p \mapsto e = \iota_*(T_pN)$.

The reader will note that we never assumed that $\mathcal{I}$ is a linear Pfaffian system. Moreover, we never prolonged the EDS; that is, we never built an ideal $\mathcal{I}^{(1)}$ on $M^{(1)}$ using the contact system $\mathcal{J}$. Instead we are working with the tautological bundle W per Remark 2.29.

As you read this part, compare it to [**IL03**, Section 4.6] and [**BCG$^+$90**, Chapter V].

4. The Characteristic Variety

The original motivation for the characteristic variety is to see where the initial-value problem becomes ambiguous.

4(a). via Polar Extension. For an integral element $e' \in \mathrm{Var}_{n-1}(TM)$, we consider its space[12] of integral extensions, called the *polar space*,

$$H(e') = \{v \ : \ e = e' + \langle v \rangle \in \mathrm{Var}_n(\mathcal{I})\} \subset TM \tag{4.1}$$

and the *polar equations* comprise its annihilator,

$$H^\perp(e') = \{e' \lrcorner \varphi \ : \ \varphi \in \mathcal{I}_n\} \subset T^*M. \tag{4.2}$$

The *polar rank* is $r(e') = \dim H(e') - \dim e' - 1$. If $r(e') = -1$, then e' admits no extensions. If $r(e') = 0$, then e' admits a unique extension to some $e \in \mathrm{Var}_n(\mathcal{I})$.

The case of interest is $r(e') > 0$, meaning that e' admits many extensions, so the initial value problem from e' to e is ambiguous. For any $e \in M^{(1)}$, we can identify a hyperplane $e' \in \mathrm{Gr}_{n-1}(e)$ with $\xi \in \mathbb{P}e^*$ via $e' = \ker \xi$. Because $e \in M^{(1)} \subset \mathrm{Gr}_n(TM)$ where n is the maximal dimension of integral elements

[12] The polar space is a vector space thanks to the assumption that $\mathcal{I}_n$ is a finitely-generated $C^\infty(M)$-module, because that assumption implies that the polar equations over $p \in M$ are a linear subspace of T^*_pM.

of $\mathcal{I}$, the function r cannot be positive on an open set of $\mathbb{P}e^*$, so the case $r(e') > 0$ is a closed condition. Moreover, the function $r : \mathbb{P}e^* \to \mathbb{N}$ is the rank of a linear system of equations, so it defines a Zariski-closed projective algebraic variety. We choose to study that algebraic variety over $\mathbb{C}$. Hence, the typical definition of the *characteristic variety of* e is

$$\Xi_e = \{\xi \in \mathbb{P}e^* \otimes \mathbb{C} : r(\xi^\perp) > 0\} \subset \gamma_e^*. \tag{4.3}$$

Since we wish to study the ambiguity of the initial-value problem, we want to assign multiplicity to each $\xi \in \Xi_e$, based somehow on the structure of the space $H(\xi^\perp)$ and related to $r(\xi^\perp)$. This definition is refined in Section 4(b).

4(b). via Rank-1 Incidence. Section 2(b) provides another interpretation of the initial-value problem that is much more convenient than (4.3).

Fix $e \in M^{(1)}$, and suppose that both e and $\tilde{e}$ are integral extensions of $e' = \ker \xi$ for some $\xi \in e^*$. That is, $\tilde{e} \in \mathrm{Pol}_1(e)$ and $\xi \in \Xi_e$. Hence, Lemma 2.7 yields a particular rank-1 projective homomorphism $w \otimes \xi \in W \otimes V^*$. Because $H(e')$ is a vector space, Lemma 2.10 allows us to assume that $\tilde{e}$ lies near e in the open connected set $M^{(1)}$ of $\mathrm{Var}_n(\mathcal{I})$. Therefore, any such $w \otimes \xi$ lies in the tableau $A_e \subset W \otimes V^*$.

On the other hand, for fixed e and ξ, there may be various distinct $\tilde{e}$ yielding linearly independent w. With Figure 6 in mind, it is easy to see that $\dim\{w \in T_pM/e \ : \ w \otimes \xi \in A\}$ equals $r(\xi^\perp)$.

Recall the rank-one ideal $\mathscr{R}$ from Section 1. Here it applies to vector bundles. The rank-one subvariety of the tableau is the set[13]

$$\mathscr{C} = A \cap \mathrm{Var}\,\mathscr{R} = A \cap \{w \otimes \xi \ : \ w \in W,\ \xi \in V^*\}. \tag{4.4}$$

As a set, the *characteristic variety* Ξ is the projection of $\mathscr{C}$ to V^*. More precisely, Ξ is the *scheme*[14] defined by the *characteristic ideal* $\mathscr{M}$ on V^* that is obtained from the rank-one ideal on $W \otimes V^*$ in the following way: For any $\xi \in V^*$, define $\sigma_\xi : W \to H^1$ by $\sigma_\xi(w) = \sigma(w \otimes \xi)$. Note that $\dim \ker \sigma_\xi = r(\xi^\perp)$. Then $\mathscr{C}$ is the incidence correspondence[15] of Ξ for the symbol map σ_ξ. See Figure 12.

$$\begin{tikzcd} & \mathscr{C} \arrow[dl, dashed] \arrow[dr] & \\ Gr_\bullet(W) & & \Xi \end{tikzcd} \qquad \begin{tikzcd} & \{w \otimes \xi\} \arrow[dl, dashed] \arrow[dr] & \\ \ker \sigma_\xi & & \xi \end{tikzcd}$$

FIGURE 12. The rank-one variety $\mathscr{C}$ is the incidence correspondence for the characteristic scheme Ξ.

[13] The variety $\mathcal{C}$ is really just a set, not a scheme. Any matrix is either rank-one or not. Any matrix is either in the tableau or not.

[14] We must study Ξ along with its various components and multiplicities, so it is better to think of it as a scheme or ideal than as a simple-minded variety.

[15] For more background on the utility of incidence correspondences in algebraic geometry, I recommend the lecture series [Har13], which I had the pleasure of attending during my time at Fordham University.

This interpretation is *amazing*. Suddenly, two simple ideas—tableaux of matrices and rank-one matrices—come together to give a concise description of the most subtle structure in classical PDE theory.

4(c). Example: The Wave Equation. Consider the PDE $u_{11}+u_{22} = u_{00}$. To do this, we consider the manifold $M = \mathbb{R}^{3+1+3} = \mathbb{J}^1(\mathbb{R}^3, \mathbb{R})$ with coordinates $(x^1, x^2, x^3, u, p_1, p_2, p_3)$. Consider the

We work on the space $M \cong \mathbb{R}^{3+1+3}$ with coordinates $(x_1, x_2, x_3, u, p_1, p_2, p_3)$. The corresponding exterior differential system is generated by

$$\text{(4.5)} \qquad \mathrm{d}\begin{bmatrix}\theta^0\\ \theta^1\\ \theta^2\\ \theta^3\end{bmatrix} \equiv \begin{bmatrix} 0 & 0 & 0\\ \pi^1_1 & \pi^1_2 & \pi^1_3\\ \pi^2_1 & \pi^2_2 & \pi^2_3\\ \pi^3_1 & \pi^3_2 & \pi^3_3\end{bmatrix} \wedge \begin{bmatrix}\omega^1\\ \omega^2\\ \omega^3\end{bmatrix} \quad \mathrm{mod}\ \{\theta^0, \theta^1, \theta^2, \theta^3\}$$

where $\pi^1_2 = \pi^2_1$, $\pi^1_3 = \pi^3_1$, $\pi^2_3 = \pi^3_2$, and $\pi^3_3 = \pi^1_1 + \pi^2_2$.

Changing bases, this tableau is equivalent to an endovolutive one of the form

$$\text{(4.6)} \qquad (\pi^a_i) = \begin{bmatrix} a_0 & a_3 & a_4\\ a_1 & a_4 & a_2 + a_3\\ a_2 & a_0 & a_1\end{bmatrix}$$

Or in block form

$$\text{(4.7)} \qquad (\mathrm{B}^\lambda_i) = \begin{bmatrix} \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix} & \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\\ \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} & \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} & \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\end{bmatrix}$$

Note that

$$\text{(4.8)} \qquad \begin{aligned} &\mathrm{B}^1_2\,\mathrm{B}^2_3 - \mathrm{B}^1_3\,\mathrm{B}^2_3\\ &= \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\\ &= \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix} - \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}; \end{aligned}$$

in particular, rows $a > s_3 = 0$ are all zero, so the system is involutive by Theorem 3.16.

The rank-one portion cone is

$$\begin{aligned} a_0a_4 - a_1a_3 \\ a_0a_0 - a_2a_3 \\ a_0a_1 - a_2a_4 \\ a_1a_1 - a_2a_2 - a_2a_3 \\ a_3a_1 - a_0a_4 \\ a_3a_2 + a_3a_3 - a_4a_4 \\ a_4a_1 - a_0a_2 - a_0a_3 \end{aligned} \tag{4.9}$$

After a simple change of basis, this becomes the example (1.2) – (1.4).

5. Guillemin Normal Form and Eigenvalues

In this section, we reinterpret $\mathscr{C}$ and Ξ as properties of the endomorphisms B_i^λ. Our main computation tool is the structure of an endovolutive tableau discussed in Section 1(c), where W and V and A are now the projective bundles over $M^{(1)}$.

The incidence correspondence of Figure 12 is rephrased in Lemma 5.1.

Lemma 5.1. *If $\xi \in \Xi$, $v \in V$, and $w \in \ker \sigma_\xi \subset W$, then*

$$\mathrm{B}(\xi)(v)w = \xi(v)w. \tag{5.2}$$

In particular, w is an eigenvector of $\mathrm{B}(\xi)(v)$ for all v.

Proof. Set $\pi = w \otimes \xi \in \mathscr{C} \subset A$, so $\pi_i^a = w^a\xi_i$ for all a, i, and this π must satisfy the symbol relations (1.9). In particular, $w^a\xi_i = B_{k,b}^{a,\lambda} w^b \xi_\lambda$ for $a > s_i$. Therefore

$$\begin{aligned} \mathrm{B}(\xi)(v)w &= \sum_{a \leq s_i} \xi_i u^i w^a z_a + \sum_{a > s_i} B_{i,b}^{a,\lambda} w^b \xi_\lambda u^i z_a \\ &= \sum_{a \leq s_i} \xi_i u^i w^a z_a + \sum_{a > s_i} \xi_i u^i w^a z_a \\ &= \sum_{a,i} \xi_i u^i w^a z_a = \xi(v)w. \end{aligned} \tag{5.3}$$

(Here we see the utility of including the first summand in Equation (1.10).) □

Lemma 5.4 provides a sort of converse.

Lemma 5.4. *Suppose that A is an endovolutive tableau. Fix $\varphi \in U^*$ and suppose that $w \in \mathbf{W}^-(\varphi)$ such that w is an eigenvector of $\mathrm{B}(\varphi)(v)$ for every $v \in V$. Then there is a $\xi \in \Xi$ over $\varphi \in U^*$ such that $w \in \mathbf{W}^1(\varphi)$, so $w \otimes \xi \in A$.*

Proof. For each $v \in V$, let $\xi(v)$ denote the eigenvalue corresponding to v, so that $\xi(v)w = \mathrm{B}(\varphi)(v)w$. Because $\mathrm{B}(\varphi)(v)w$ is linear in v, so is

$\xi(v)$. Then $\xi = \xi_i u^i \in V^*$. Therefore, $\mathrm{B}(\varphi)(\cdot)w = w \otimes \xi$. In particular, the rank-one condition implies that

$$\sum_{\lambda \leq \mu} \varphi_\lambda \, \mathrm{B}^\lambda_\mu \, w = \xi_\mu w = \sum_{\lambda \leq \mu} \xi_\lambda \, \mathrm{B}^\lambda_\mu \, w, \ \forall \mu \leq \ell. \tag{5.5}$$

This is the same expression as in (1.13), so by comparing recursively over $\mu = 1, 2, \ldots, \ell$, we see that $\xi_\lambda = \varphi_\lambda$ for all λ, so $w \in \mathbf{W}^1(\varphi) \subset \mathbf{W}^-(\varphi)$. $\square$

Lemma 5.4 deserves a warning: There may be multiple ξ over the same φ, for perhaps there are different $w \in \mathbf{W}^-(\varphi)$ admitting different sequences of eigen*values* ξ_ϱ, for $\varrho > \ell$, associated to the same φ. It is also not (yet) clear that a mutual eigenvector w exists for every such φ.

Overall, it is clear that there is a some relationship between the eigenvalues of B^λ_i and the characteristic variety of an endovolutive tableau A. This relationship is made precise for involutive tableaux using a result from [**Gui68**].

Theorem 5.6 (Guillemin normal form). *Suppose that A is involutive. For every $\varphi \in U^*$ and $v \in V$, the restricted homomorphism $\mathrm{B}(\varphi)(v)|_{\mathbf{W}^1(\varphi)}$ is an endomorphism of $\mathbf{W}^1(\varphi)$. Moreover, for all $v, \tilde{v} \in V$,*

$$[\mathrm{B}(\varphi)(v), \mathrm{B}(\varphi)(\tilde{v})]\Big|_{\mathbf{W}^1(\varphi)} = 0. \tag{5.7}$$

Compare Theorem 5.6 to Lemma 4.1 in [**Gui68**] and Proposition 6.3 in Chapter VIII of [**BCG$^+$90**]. Theorem 5.6 is known as *Guillemin normal form* because it implies that the family of homomorphisms $\mathrm{B}(\varphi)(\cdot)$ can be placed in simultaneous Jordan normal form on $\mathbf{W}^1(\varphi)$. It is the "normal form" alluded to in Section 1(b). We defer the proof to Section 6 so we may first see its important consequences.

Corollary 5.8. *If A is involutive, then for each $\varphi \in U^*$, there exists some w satisfying the hypotheses of Lemma 5.4. That is, the projection map $\Xi \to U^*$ is onto. In particular, if A is nontrivial and involutive, then Ξ is nonempty.*

Proof. Because we are working over $\mathbb{C}$, the commutativity condition (5.7) guarantees that common eigenvectors exist for $\{\mathrm{B}(\varphi)(v) \ : \ v \in V\}$. $\square$

Lemma 5.9. *Suppose that A is an involutive tableau. Then the map of projective varieties induced by $\Xi \to U^*$ is a* finite branched cover. *In particular, we have the affine dimensions $\dim \Xi = \dim U^* = \ell$.*

Proof. Fix $\varphi \in U^*$. The set of ξ over φ is nonempty by Corollary 5.8. If it were true that the set of ξ projecting to a particular φ were infinite, then the parameter ξ_i would take infinitely many values in some expression of the form

$$\det \left(\sum_\lambda \varphi_\lambda \, \mathrm{B}^\lambda_i - \xi_i I \right) = 0. \tag{5.10}$$

But, the matrix $\sum_\lambda \varphi_\lambda \, \mathrm{B}^\lambda_i \in \mathrm{End}(\mathbf{W}^-_1)$ can have at most s_1 eigenvalues. $\square$

Here we arrive at an easy[16] proof of the main theorem regarding the structure of Ξ.

THEOREM 5.11. *If A is involutive, then $\dim \Xi = \ell - 1$ and $\deg \Xi = s_\ell$.*

PROOF. We work in endovolutive coordinates. From Lemma 5.9, we already know that $\dim \Xi = \ell - 1$.

Fix a generic point $\xi \in \Xi$ over $\varphi \in U^*$. We must determine the degree of the condition $\mathscr{C}_\xi \neq 0$. Note that $\mathscr{C}_\xi$ must be a subvariety of $\mathbf{W}^1(\varphi) \otimes \xi$, and $\mathbf{W}^1(\varphi)$ is a linear subspace of W, so the degree of Ξ is the degree of some condition on $\mathbf{W}^1(\varphi)$.

By Lemma 5.1 and (4.4), the condition that $\mathscr{C}_\xi$ is nontrivial is precisely the condition that

$$\det\left(\sum_\lambda \xi_\lambda \, \mathrm{B}^\lambda_i - \xi_i I\right) = 0, \ \forall i. \tag{5.12}$$

Since we may restrict our attention to $\mathbf{W}^1(\varphi)\otimes\xi$, only these terms contribute to the non-linear part of the ideal:

$$\det\left(\sum_\lambda \xi_\lambda \, \mathrm{B}^\lambda_\varrho - \xi_\varrho I\right) = 0, \ \forall \varrho > \ell. \tag{5.13}$$

or, without coordinates,

$$\det\left(\mathrm{B}(\xi)(v) - \xi(v) I\right) = 0, \ \forall v \in (U^*)^\perp. \tag{5.14}$$

For a particular v, this is the characteristic polynomial of $\mathrm{B}(\xi)(v)$ as an endomorphism of $\mathbf{W}^1(\varphi)$. By involutivity and Theorem 5.6, all $\mathrm{B}(\xi)(v)$ for $v \in (U^*)^\perp$ admit the same factorization type for their respective characteristic polynomials, so it does not matter which v we consider. By definition, the characteristic polynomial of $\mathrm{B}(\xi)(v)|_{\mathbf{W}^1(\varphi)}$ has degree $\dim \mathbf{W}^1(\varphi)$. Therefore, $\deg \Xi = s_\ell$ follows from Lemma 1.15. $\square$

Theorems 5.6 and 5.11 provide a powerful interpretation of the form of an involutive tableau seen in Theorem 3.16 and Figure 3; the first ℓ columns represent a projection of Ξ, as in Lemma 5.9, and the rank-one incidence correspondence in Figure 12 is precisely the eigenvector condition on the appropriate subspaces. It is peculiar and interesting that these results were discovered in the opposite order historically.

6. Results of Guillemin and Quillen

Guillemin's proof of Theorem 5.6 made use of two results derived from Quillen's thesis [Qui64]. In this section, we see how easy these results become using Theorem 3.16. (Note that Theorem 3.16 and Theorem 5.6 are *not* equivalent. It is easy to construct endovolutive tableaux that satisfy (5.7) but are not involutive.)

[16] It is easy in the sense that we do not need to invoke the Riemann-Roch theorem, as we have the explicit polynomials of $\mathscr{M}$ in hand, and they are recognizable as the familiar eigenvector equations.

Recall the Spencer cohomology groups from Section 3(b). For any $\varphi \in V^*$, wedging by φ gives a map $W \otimes \wedge^p V^* \to W \otimes \wedge^{p+1} V^*$. This induces a map on the quotient spaces, $H^p(A) \to H^{p+1}(A)$.

Theorem 6.1 (Quillen's Exactness Theorem). *Suppose A is an involutive tableau, and that $\varphi \notin \Xi_A$. Then the sequence of maps by $\wedge\varphi$,*

$$0 \to A \to H^1(A) \to H^2(A) \to \cdots \to H^n(A) \to 0,$$

is exact.

In [Qui64], this theorem is proven using enormous commutative diagrams. In our context, with Theorem 3.16 in hand, we can prove an easy version of Quillen's result, in the form of Lemma 6.3.

Lemma 6.3 is a consequence of 6.2, which for us is an easy corollary of Theorem 3.16. This corollary is called Theorem A in [Gui68], where it was proved using a large diagram chase using Quillen's exactness theorem 6.1.

Corollary 6.2 (Quillen, Guillemin). *If A is involutive, then $A|_U$ is involutive, and the natural map between prolongations $A^{(1)} \to (A|_U)^{(1)}$ is bijective.*

Proof. The first part is an immediate consequence of Theorem 3.16, as the quadratic condition still holds if the range of indices λ, μ, i, j is truncated at ℓ (or greater). In particular, the generators $(\pi^a_\mu)_{a \leq s_\mu}$ of A are preserved.

The second part is similarly immediate, using the proof of Theorem 3.16 given in [Smi15]: the contact relation $\pi^a_\mu = Z^a_{\mu,i} u^i$ for $a \leq s_\lambda$ gives coordinates $Z^a_{\mu,i}$ to the prolongation $A^{(1)} \subset A \otimes V^*$, and the $s_1 + 2s_2 + \cdots + \ell s_\ell$ independent generators are precisely those $Z^a_{\mu,\lambda}$ with $a \leq s_\mu$ and $\lambda \leq \mu \leq \ell$. Since they involve no indices $i > \ell$, these generators remain independent when the range of indices is truncated at ℓ. □

Now we come to our simplified version of Theorem 6.1. Compare Lemma 6.3 to the exact sequence $(3.4)_2$ in [Gui68].

Lemma 6.3. *Recall that Y^* is a complement to $U^* \subset V^*$, so that $V^* = U^* \oplus Y^*$. For A involutive, the sequence*

$$0 \to W \otimes S^2 Y^* \to H^1 \otimes Y^* \xrightarrow{\delta} H^2$$

is exact.

Proof. This proof is just an explicit description of the maps in a basis and an application of Corollary 6.2. Let (u^i) be a basis for V^* such that (u^λ) is a basis for U^* and (u^ϱ) is a basis for Y^*, using the index convention (1.8) from Section 1.

The sequence makes sense because we can split the Spencer sequence (3.11) as $W \otimes V^* = A \oplus H^1$ by identifying the space H^1 with $\{\sum_{a > s_i} \pi^a_i (z_a \otimes u^i)\} \subset W \otimes V^*$, which is the space spanned by the unshaded entries in Figure 1. Using this identification, two elements $\sum_{a>s_i} \pi^a_i (z_a \otimes u^i)$ and $\sum_{a>s_i} \hat{\pi}^a_i (z_a \otimes u^i)$ of $W \otimes V^*$ are equivalent in H^1 if and only if $\pi^a_i - \hat{\pi}^a_i = \sum_{b \leq s_\lambda} B^{a,\lambda}_{i,b} z^b_i$ for some $\{z^a_i : a \leq s_i\}$, the shaded entries in Figure 1. In other words, the projection

$W \otimes V^* \to H^1$ is defined by (1.9), and the projection $W \otimes V^* \to A$ is defined by the projection onto the orange generator components in Figure 1, those π^a_λ with $a \leq s_\lambda$.

Since $s_\varrho = 0$ for all $\varrho > \ell$, the inclusion $W \otimes Y^* \subset W \otimes V^*$ is an inclusion $W \otimes Y^* \subset H^1$. Hence, the inclusion is understood as

$$W \otimes S^2 Y^* \subset (W \otimes Y^*) \otimes Y^* \subset H^1 \otimes Y^*. \tag{6.4}$$

An element of $H^1 \otimes Y^*$ is written in $W \otimes V^* \otimes Y^*$ as

$$P = \sum_{a > s_\lambda} P^a_{\lambda,\varsigma}(z_a \otimes u^\lambda \otimes u^\varsigma) + \sum_{a>0} P^a_{\varrho,\varsigma}(z_a \otimes u^\varrho \otimes u^\varsigma). \tag{6.5}$$

The image $\delta(H^1 \otimes Y^*)$ in H^2 is

$$\delta(H^1 \otimes V^*) \subset \delta(W \otimes V^* \otimes V^*) \subset W \otimes \wedge^2 V^*, \tag{6.6}$$

so $\delta P \in W \otimes \wedge^2 V^*$ is of the form

$$\delta P = \sum_{a > s_\lambda} P^a_{\lambda,\varsigma}(z_a \otimes u^\lambda \wedge u^\varsigma) + \sum_{a>0} \frac{1}{2}\left(P^a_{\varrho,\varsigma} - P^a_{\varsigma,\varrho}\right)(z_a \otimes u^\varrho \wedge u^\varsigma). \tag{6.7}$$

Recall that $H^2 = \frac{W \otimes \wedge^2 V^*}{\delta_\sigma(A \otimes V^*)}$. So, $\delta P \equiv 0$ in H^2 if and only if there is some $T \in A \otimes V^*$ such that $\delta_\sigma(T) = \delta(P)$ in $W \otimes \wedge^2 V^*$. Looking at (6.7), it is apparent that such T must have $\delta_\sigma(T|_U) = 0$, as $\delta(P)$ has no $U^* \wedge U^*$ terms. By involutivity and Corollary 6.2, we consider the involutive tableau

$$0 \to A|_U \to W \otimes U^* \xrightarrow{\sigma|_U} H^1_U \to 0 \tag{6.8}$$

with prolongation

$$0 \to (A|_U)^{(1)} \to A|_U \otimes U^* \xrightarrow{\delta_\sigma|_U} W \otimes \wedge^2 U^* \to H^2_U \to 0. \tag{6.9}$$

Therefore, $T|_U \in A|_U \otimes U^*$ lies in the kernel of $\delta_\sigma|_U$, so $T|_U \in (A|_U)^{(1)}$. Therefore, Corollary 6.2 tells us $T \in A^{(1)}$. That is, $\delta(P) \equiv 0 \in H^2$ if and only if $\delta(P) = \delta_\sigma(T) = 0$.

Therefore, $\delta(P) \equiv 0 \in H^2$ if and only if $P^a_{\lambda,\varsigma} = 0$ and $P^a_{\varrho,\varsigma} = P^a_{\varsigma,\varrho}$ on these index ranges. This occurs if and only if $P = P^a_{\varrho,\varsigma}(z_a \otimes u^\varrho \otimes u^\varsigma)$, meaning $P \in W \otimes S^2 Y^*$. ∎

We are ready to prove Theorem 5.6. The structure of the proof is identical to the original proof in [Gui68].

Proof of Theorem 5.6. Suppose that $w \in \mathbf{W}^1(\varphi)$, so that $\pi = \mathrm{B}(\varphi)(\cdot)w = w \otimes \varphi + J$ for some $J \in W \otimes Y^*$ with $J_\varrho = J^a_\varrho z_a \in \mathbf{W}^-(\varphi)$ for all ϱ. First, we must show that the span of the columns J_ϱ of J lies in $\mathbf{W}^1(\varphi)$.

Consider the element $-J \otimes \varphi = -J^a_\varrho \varphi_\lambda(z_a \otimes u^\lambda \otimes u^\varrho) \in H^1 \otimes Y^*$. Because $z \otimes \varphi + J \in A$, it must be that $z \otimes \varphi \otimes \varphi \in W \otimes V^* \otimes V^*$ represents the same point in $H^1 \otimes Y^*$. So, we can compute

$$-J^a_\varrho \varphi_\lambda(z_a \otimes u^\lambda \wedge u^\varrho) \equiv z \otimes \varphi \wedge \varphi = 0 \in H^2 \tag{6.10}$$

By Corollary 6.2, there exists $Q = Q^a_{\varrho,\varsigma}(z_a \otimes u^\varsigma \otimes u^\varrho) \in W \otimes S^2 Y^*$ such that $-J \otimes \varphi - Q \in A \otimes Y^*$. That is, writing $Q_\varrho = Q^a_{\varrho,\varsigma}(z_a \otimes u^\varsigma) \in W \otimes U^*$, we have $J_\varrho \otimes \varphi + Q_\varrho \in A$ for all ϱ, meaning $J_\varrho \in \mathbf{W}^1(\varphi)$ for all ϱ. Therefore, for any $v \in V$, we have $\mathrm{B}(\varphi)(v)z = \varphi(v)z + J(v) \in \mathbf{W}^1(\varphi)$.

Now, mapping again, $\mathrm{B}(\varphi)(\cdot)J_\varrho = J_\varrho \otimes \varphi + Q_\varrho$, so $\mathrm{B}(\varphi)(u_\varsigma)J_\varrho = Q_{\varrho,\varsigma}$, which is already known to be symmetric in ϱ, ς. Therefore,

$$\begin{aligned}\mathrm{B}(\varphi)(\tilde{v})\,\mathrm{B}(\varphi)(v)z &= \mathrm{B}(\varphi)(\tilde{v})\left(\varphi(v)z + J(v)\right)\\ &= \varphi(v)\,\mathrm{B}(\varphi)(\tilde{v})z + u^\varrho(v)\,\mathrm{B}(\varphi)(\tilde{v})J_\varrho\\ &= \varphi(v)\left(\varphi(\tilde{v})z + J(\tilde{v})\right) + u^\varrho(v)\left(\varphi(\tilde{v})J_\varrho + Q_\varrho(\tilde{v})\right)\\ &= \varphi(v)\varphi(\tilde{v})z + \varphi(v)J(\tilde{v}) + \varphi(\tilde{v})J(v) + Q(v,\tilde{v}).\end{aligned} \tag{6.11}$$

This is symmetric in $v, \tilde{v}$, giving the commutativity condition (5.7) $\square$

It is interesting to see the inversion of logic that happened here. In the original literature, the overall implications are

$$6.1 \to 6.3 \to 6.2 \to 5.6.$$

But, the arguments here give the overall implication

$$3.16 \to 6.2 \to 6.3 \to 5.6.$$

However, we can write a shorter proof of Theorem 5.6 that relies Theorem 3.16 more directly, avoiding the general results of Quillen. For motivation, consider the following trivial corollary of Theorem 3.16 that is obtained by setting $\lambda = \mu$:

Corollary 6.12. *Under the assumptions of Theorem 3.16, $\mathrm{B}(u^\lambda)(v)$ is an endomorphism of $\mathbf{W}^-(u^\lambda)$ such that for all $v, \tilde{v} \in (U^*)^\perp$,*

$$[\mathrm{B}(u^\lambda)(v), \mathrm{B}(u^\lambda)(\tilde{v})] = 0.$$

Alternate Proof of Theorem 5.6. Fix $\varphi \in U^*$, and suppose that $w \in \mathbf{W}^1(\varphi)$. We must verify that all maps $\mathrm{B}(\varphi)(v)$ preserve $\mathbf{W}^1(\varphi)$ and that they commute. Note that the definition of $\mathbf{W}^1(\varphi)$ in Equation 1.14 depends on the choice of subspace U^* but not on its basis, so we may verify these conditions using any basis we like.

First a trivial case: if it happens that $\varphi \in \Xi \cap U^*$, then $\mathrm{B}(\varphi)(v)w = \varphi(v)w \in \mathbf{W}^1(\varphi)$ is a rescaling, and it is immediate that $[\mathrm{B}(\varphi)(v), \mathrm{B}(\varphi)(\tilde{v})] = 0$.

Otherwise, we have $\varphi \not\in \Xi$. Then we may choose a regular basis of V^* in which $\varphi = u^1$. Moreover, we may use that basis to construct an endovolutive basis of W. By Corollary 6.12, it suffices to prove in this basis that $\mathbf{W}^1(u^1)$ is preserved by every $\mathrm{B}(u^1)(v)$. Write $\mathrm{B}(\varphi)(\cdot)w = w \otimes u^1 + J$, and examine (1.13) on a column J_ϱ of J. For each $\mu = 1, \ldots, \ell$, we must verify

$$0 = \left(\mathrm{B}^1_\mu - \delta^1_\mu I\right) J_\varrho = \left(\mathrm{B}^1_\mu - \delta^1_\mu I\right) \mathrm{B}^1_\varrho\, w = \left(\mathrm{B}^1_\mu\, \mathrm{B}^1_\varrho - \delta^1_\mu\, \mathrm{B}^1_\varrho\right) w \tag{6.13}$$

If $\mu = 1$, then this is immediately 0, since $\mathrm{B}^1_1 = I_{s_1}$.

If $\mu \neq 1$, then we are checking $\mathrm{B}^1_\mu\, \mathrm{B}^1_\varrho\, w$. Note that $\mathrm{B}^1_\mu w = 0$, since $\mathrm{B}(\varphi)(\cdot)w = w \otimes \varphi + J$. Moreover, by Theorem 3.16, we have

$$0 = \left(\mathrm{B}^1_\mu\, \mathrm{B}^1_\varrho - \mathrm{B}^1_\varrho\, \mathrm{B}^1_\mu\right)^a_b w^b = \left(\mathrm{B}^1_\mu\, \mathrm{B}^1_\varrho\right)^a_b w^b \tag{6.14}$$

for $a > s_\mu$. Therefore, $\mathrm{B}^1_\mu \mathrm{B}^1_\varrho$ lies in $\mathbf{W}^-(\mu)$. On the other hand, note that the output of B^1_μ lies in $\mathbf{W}^+_\mu$ by the construction of the maps B^λ_μ from the reduced symbol in Section 1(c). Combining these, we see that $\mathrm{B}^1_\mu \mathrm{B}^1_\varrho w$ lies in $\mathbf{W}^-_\mu \cap \mathbf{W}^+_\mu = 0$.

Hence, the space $\mathbf{W}^1(\varphi)$ is preserved by $\mathrm{B}(\varphi)(v)$ for all v. By Corollary 6.12, they commute. □

On the theoretical side, it would be interesting to see how many of the hard classical theorems in the subject can be re-proven with elementary techniques. (Existing references such as [BCG+90] present elementary proofs only in the case of rectangular tableaux.) Specifically, the proof of Lemma 6.3 suggests an elementary proof of Quillen's exactness theorem. The other hard theorem is the integrability of the characteristic variety, and a proof of that theorem using Guillemin's original formulation is the subject of [GQS70]. That result was applied immediately to study primitive Lie pseudogroups.

7. Prolongation

How does the characteristic scheme change under prolongation? The short answer is that it does not!

Recall that $A^{(1)}$ is a tableau within $A \otimes V^*$. An element of $A^{(1)}$ is $P \in A \otimes V^*$. Using any bases for V, W, A, we may write P as $P^a_{i,j} z_a \otimes u^i \otimes u^j$, with the additional condition that $P^a_{i,j} = P^a_{j,i}$ from (3.5). Let $\mathscr{C}^{(1)}$ denote the rank-one elements of $A^{(1)}$, and let $\Xi^{(1)}$ denote its projection to V^*, as in Section 4(b).

This does not depend on endovolutivity or involutivity.

Theorem 7.1. *If $\pi \otimes \xi \in \mathscr{C}^{(1)}$, then $\pi = w \otimes \xi \in \mathscr{C}$ for some $w \in \ker \sigma_\xi$. Conversely, if $w \otimes \xi \in \mathscr{C}$, then $(w \otimes \xi) \otimes \xi \in \mathscr{C}^{(1)}$. In particular, $\Xi \cong \Xi^{(1)}$ as schemes.*

Proof. Suppose that $\pi \otimes \xi \in \mathscr{C}^{(1)}$ for some $\pi \in A$ and $\xi \in V^*$. That is, $P \in A^{(1)}$ and $P = \pi \otimes \xi$, so $P^a_{i,j} = \pi^a_i \xi_j$, and $\pi^a_i \xi_j = \pi^a_j \xi_i$ for all a, i, j.

Let $\underline{\lambda}$ be the minimum index such that $\xi_{\underline{\lambda}} \neq 0$. Then $\pi^a_{\underline{\lambda}} \xi_i = \pi^a_i \xi_{\underline{\lambda}}$, so column i of (π^a_i) is a multiple—namely $\xi_i/\xi_{\underline{\lambda}}$—of column $\underline{\lambda}$ for all i. Therefore, (π^a_i) is rank-one, and there is some w with $\pi = w \otimes \xi$. The converse is immediate. □

Remark 7.2. Theorem 7.1 is sometimes used as a method for computing the characteristic variety, as follows: Given a tableau (π^a_i) whose entries might depend on $e \in M^{(1)}$, consider $(\xi_i) \mapsto (\pi^a_i \xi_j - \pi^a_j \xi_i)$ as a map $V^* \to W \otimes \wedge^2 V^*$; that is, a map from $\mathbb{C}^n$ to $\mathbb{C}^{r\binom{n}{2}}$. For a general point in $\xi \in V^*$, this map has rank at least 1. Its rank falls to 0 if and only if $\xi \in \Xi$. But, I don't recommend it. If you have (π^a_i) in hand and want to compute 2×2 minors of something, you might as well compute the 2×2 minors of (π^a_i) itself!

8. Characteristic Sheaf

Suppose that I tell you I am thinking of an $r \times r$ complex matrix. I tell you the dimension of the vector space as well as the number of generalized eigenspaces and the dimension of each. Perhaps I even give you some relationships among the eigenvalues. Then, I ask you some questions about the matrix as a map. (In fact, virtually every question in an undergraduate linear algebra class is a variation of this game.) The *only* coordinate-invariant questions that you would be unable to answer are those that require the eigenvalues themselves. This is the utility of Jordan normal form.

The ultimate conclusion of the preceding sections is that the situation for differential systems is quite similar. The characteristic sheaf $\mathscr{M}$ knows the dimensions n, r, $(s_1, \ldots, s_n)$, as well as all of the dimensions and relationships among the mutual eigenspaces of the various symbol maps. It therefore also knows on what subspaces the symbol maps fail to commute. Expressed as the rank-one incidence correspondence, it even knows algebraic relationships among the sequences of eigenvalues (which we call ξ). Moreover, it does not care about prolongation. In summary, $\mathcal{M}$ (or $\mathcal{C}$) knows everything important about an abstract tableau A.

If this formal perspective is appealing, then one might as well dispense with matrices, bases, and differential forms and instead study $\mathscr{M}$ directly, with modern algebraic tools such as [**Eis05**].

Consider $\mathscr{M}$ as an ideal in $C^\infty(M)[u_1, \ldots, u_n]$, and consider its free resolution. The Hilbert syzygy theorem states that there is a finite free resolution that is characterized by its Hilbert polynomial $h_{\mathscr{M}}(d)$. Of course, Theorem 5.11 is reading the leading term of $h_{\mathscr{M}}(d)$!

One might ask how the involutivity of A can be detected as an algebraic property of $\mathscr{M}$. The answer is tied to Castelnuovo–Mumford regularity, which measures the growth of the Hilbert polynomial. This is equivalent to the Cartan characters in Cartan's test!

While it is not necessarily a useful *computational* tool versus exterior forms and tableaux, this perspective allows a broader view of the techniques in PDE analysis, and it suggests that a detailed further progess in the field will come through an emphasis on invariant algebraic techniques.

For more on this perspective, see [**Mal03**], [**BCG$^+$90**, Chapter VIII], and the notes by Mark Green from the 2013 conference *New Directions in Exterior Differential Systems* in Estes Park, Colorado.

Part III

Eikonal Systems

In Part II, we studied the characteristic sheaf as defined over $M^{(1)} \subset \mathrm{Gr}_n(TM)$. In this part, we turn our attention to the characteristic sheaf as pulled back to an ordinary integral manifold $\iota : N \to M$. This is where the meaning of Ξ as "directions with an ambiguous initial value problem" has clear implications for the structure of particular solutions of a differential equation.

9. General Eikonal Systems

First, let us consider the general notion of "eikonal equations" of a projective variety, without specific regard to the characteristic variety.

Consider a smooth manifold N of dimension n. The implicit function theorem says that a smooth hypersurface $H \subset N$ is defined locally by a smooth function $f : N \to \mathbb{R}$, where $T_xH = \ker \mathrm{d}f$. By the Frobenius theorem, this is equivalent to having a local smooth section φ of T^*N such that $\mathrm{d}\varphi \equiv 0 \mod \varphi$, for then φ is a rescaling of some $\mathrm{d}f$.

We can also look at the Frobenius theorem from the perspective of Cartan–Kähler theory[17], as in Theorem 3.15. To make a smooth function $f : N \to \mathbb{R}$ or a local section φ of T^*N, consider the jet space $\mathbb{J}^1(N, \mathbb{R})$, which is isomorphic to the bundle $T^*N \times \mathbb{R}$. Jet space is an open neighborhood (or local linearization) of $\mathrm{Gr}_n(N \times \mathbb{R})$ with local coordinates $(x^i, p_i, y) = (x^1, \ldots, x^n, p_1, \ldots, p_n, y)$ and a contact system $\mathcal{J}$ generated by $\Upsilon = \mathrm{d}u - p_i \mathrm{d}y^i$ and $\mathrm{d}\Upsilon$, as in Section 2(e). Any n-dimensional integral manifold of $\mathcal{J}$ on which $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \neq 0$ corresponds to a function $y = f(x^1, \ldots, x^n)$ with $p_i = \frac{\partial f}{\partial x^i}$, so we may take $\varphi = \mathrm{d}f = \frac{\partial f}{\partial x^i} \mathrm{d}x^i$. It is easy to see that the exterior differential system $\mathcal{J}$ on $T^*N \times \mathbb{R}$ has no torsion and has a "free" tableau with characters $s_1 = s_2 = \cdots = s_n = 1$. That is, integral manifolds are parametrized by 1 function of n variables—hardly a surprise!

Now, consider a projective subbundle $\Sigma_N \subset \mathbb{P}T^*N$, meaning it is defined smoothly by homogeneous functions in the local fiber variables (p_i) of T^*N. We want to know whether we can find hypersurfaces H for which $\mathrm{d}f \in \Sigma_N$ everywhere. Specifically, we want like a theorem like this:

Theorem 9.1. *Suppose that the eikonal system (defined below) of Σ_N is involutive. Then for any smooth point $\varphi \in \Sigma_{N,x}$, there is a smooth hypersurface $H \subset N$ such that $(T_xH)^\perp = \varphi$ and such that $(T_{\tilde{x}}H)^\perp$ lies in the smooth locus of $\Sigma_{N,\tilde{x}}$ for all $\tilde{x} \in H$.*

Because the section φ is not chosen *a priori*, this condition is difficult to interpret using the original Frobenius formulation of hypersurfaces; however, the formulation on $T^*N \times \mathbb{R}$ is well-suited to this theorem. Consider the inclusion $\psi : \Sigma_N \times \mathbb{R} \to \mathbb{J}^1(N, \mathbb{R})$. The *eikonal system* of Σ is the exterior differential system $\mathscr{E}(\Sigma_N) = \psi^*(\mathcal{J})$ on $\Sigma_N \times \mathbb{R}$; that is, $\mathscr{E}(\Sigma_N)$ is generated by $\psi^*(\Upsilon)$ and $\psi^*(\mathrm{d}\Upsilon)$ and has independence condition $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \neq 0$.

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[17] Although Theorem 3.15 applies as stated only in the analytic category, it can obviously be extended to the smooth category in this case. This sort of extension is explored in Section 11.

An integral manifold of $\mathscr{E}(\Sigma_N)$ corresponds to a hypersurface in N whose tangent space in annihilated by a section of Σ_N.

We do not prove involutivity of $\mathcal{E}(\Sigma_N)$ in any significant case here; it is typically extremely deep and difficult, and references are provided below. However, the situation in Theorem 9.1 has several interesting consequences and interpretations.

COROLLARY 9.2. *Let $\ell - 1$ denote the projective dimension of Σ_N. In the situation of Theorem 9.1, such hypersurfaces depend on ℓ functions of 1 variable.*

PROOF. Fix $\varphi \in \Sigma_{N,x}$. We work locally (actually, microlocally!) near φ, so we may assume N is open, connected, and simply connected, and that $T^*N = N \times \mathbb{R}^n$. Because Σ_N is smooth of affine dimension ℓ in T^*N, we may choose local coordinates $(q_1, \ldots, q_n)$ on each fiber of T^*N near such that Σ_N is defined by $q_{\ell+1} = \cdots = q_n = 0$ near φ.

For each $\lambda = 1, \ldots, \ell$, let $\sigma^\lambda \in \Sigma_{N,x}$ denote the 1-form specified as $(0, \ldots, 0, q_\lambda, 0, \ldots 0)$ in these coordinates. By Theorem 9.1, there is a local hypersurface $H_\lambda \subset N$ and a corresponding local function x^λ such that $\mathrm{d}x^\lambda \sim \sigma^\lambda$. Complete $x^1, \ldots, x^\ell$ to a local coordinate system (x^i) on N, and let p_i be the corresponding "derivative" coordinates $p_i = \frac{\partial}{\partial x^i}$ on the fiber of T^*N. Note that $p_i(\mathrm{d}x^\ell) = \delta_i^\lambda$ by construction, so Σ_N is defined by $p_{\ell+1} = \cdots = p_n = 0$. (Note that the open neighborhood of T^*N around φ may have shrunk during this process.)

Therefore, the contact system on $T^*M \times \mathbb{R}$ is generated in a neighborhood of φ by $\Upsilon = \mathrm{d}y - p_i \mathrm{d}x^i$, which pulls back to $\Sigma_N \times \mathbb{R}$ as

$$\psi^*(\Sigma_N) = \mathrm{d}y - p_\lambda \mathrm{d}x^\lambda.$$

The corresponding tableau is the space of $1 \times \ell$ with entries $\mathrm{d}p_\lambda$ for $\lambda = 1, \ldots, \ell$, so its Cartan characters are $s_1 = s_2 = \cdots = s_\ell = 1$. $\square$

It is easy to adapt the previous proof to the following corollary, which is useful for constructing coordinates in some situations, as in [**Smi14**].

COROLLARY 9.3. *For any Σ_N, let $\langle \Sigma_N \rangle$ denote its linear span, which is itself a projective subbundle of $\mathbb{P}T^*N$. If $\mathscr{E}(\Sigma_N)$ is involutive, then $\mathscr{E}(\langle \Sigma_N \rangle)$ is involutive.*

The eikonal system has several interpretations that tie together various branches of geometry. Compare Sections 9(a) and 9(b) to [**BCG$^+$90**, V§3(vi)].

9(a). Eikonal Systems as Lagrangian Geometry. The $\mathbb{R}$ term in $T^*N \times \mathbb{R}$ plays little role for the eikonal system $\mathcal{E}(N\Sigma_N)$. It is there merely to make obvious the relationship between the eikonal equations and hypersurfaces.

Instead, consider the symplectic manifold T^*N with symplectic 2-form $\mathrm{d}\Upsilon$, which is expressed in local coordinates as $\mathrm{d}\Upsilon = \mathrm{d}p_i \wedge \mathrm{d}x^i$ according to

Darboux's theorem. The *Lagrangian Grassmannian* $LG(N)$ is the bundle over T^*N whose fiber is all the Legendrian n-planes

$$LG_\varphi(N) = \{e \in \mathrm{Gr}_n(T_\varphi T^*N) \ :\ \mathrm{d}\Upsilon|_e = 0\}. \tag{9.4}$$

Each fiber is isomorphic to the homogeneous space $LG(n, 2n)$, which is the variety of n-planes in $\mathbb{R}[x^1, \ldots, x^n, p_1, \ldots p_n]$ on which $\mathrm{d}p_i \wedge \mathrm{d}x^i = 0$. If we consider a plane $e \in LG(n, 2n)$ for which $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \neq 0$, then $\mathrm{d}p_i = P_{i,j}(e)\mathrm{d}x^i$ on e with $P_{i,j} = P_{j,i}$. Hence, the non-vertical open neighborhood of $LG(n, 2n)$ is identified with the space of symmetric matrices, $\mathrm{Sym}^2(\mathbb{R}^n)$.

Suppose the affine subvariety $\Sigma_N \subset T^*N$ is defined smoothly by homogeneous functions in the local fiber variables (p_i) of T^*N. From this perspective, the eikonal system $\mathcal{E}(\Sigma_N)$ is measuring the intersection of $\mathrm{Gr}_n(T_\varphi\Sigma_N)$ with $LG_\varphi(N)$ for all $\varphi \in \Sigma_N$.

Corollary 9.5. *The eikonal system $\mathcal{E}(\Sigma_N)$ is involutive if and only if there are local coordinates of T^*N near $\varphi \in \Sigma_N$ in which the non-vertical open set in $\mathrm{Gr}_n(T_\varphi\Sigma_N) \cap LG_\varphi(N)$ is described as the $n \times n$ symmetric matrices $P_{i,j}(e)$ that vanish outside the upper-left $\ell \times \ell$ part.*

Proof. If the eikonal system $\mathscr{E}(\Sigma_N)$ is involutive, then we may construct coordinates as in Corollary 9.2 such that Σ_N is defined by $p_\varrho = 0$ for all $\varrho > \ell$, so $T_\varphi\Sigma_N$ is defined by $\mathrm{d}p_\varrho = 0$ for all $\varrho > \ell$. In such coordinates, the open neighborhood of the Lagrangian Grassmannian takes the block form

$$\begin{pmatrix}\mathrm{d}p_\lambda \\ \mathrm{d}p_\varrho\end{pmatrix}|_e = \begin{pmatrix}P_{\lambda,\mu}(e) & P_{\lambda,\varsigma}(e) \\ P_{\varrho,\mu}(e) & P_{\varrho,\varsigma}(e)\end{pmatrix}\begin{pmatrix}\mathrm{d}x^\mu \\ \mathrm{d}x^\varsigma\end{pmatrix}|_e, \text{ such that } P_{i,j} = P_{j,i}, \tag{9.6}$$

using our index convention (1.8) from Section 1. The condition $e \in T\Sigma_N$ implies $\mathrm{d}p_\varrho = 0$, so the lower blocks are zero. The matrix is symmetric, so the upper-right block is zero.

Conversely, suppose such coordinates exist. Then $T\Sigma_N$ satisfies the closed 1-forms $\mathrm{d}p_\varrho = 0$, and the dimensions match, so Σ_N satisfies $p_\varrho =$ constant. Since the equations defining Σ_N are homogeneous, it must be $p_\varrho = 0$. Using these coordinates for $T^*N \times \mathbb{R}$ and $\mathcal{J}$ yields $\psi^*(\Upsilon) = \mathrm{d}y - p_\lambda \mathrm{d}x^\lambda$, as in Corollary 9.2, which is involutive with the correct Cartan characters and gives the desired hypersurfaces in Theorem 9.1. ∎

Compare this to Proposition 3.22 in [**BCG$^+$90**, Chapter V]. For more symplectic and Lagrangian geometry, see [**Bry93**].

9(b). Eikonal Systems as Poisson Brackets. If T^*N describes the physical state of a system, a function $F : T^*N \to \mathbb{R}$ is called an *observable* [**SW86**]. The *Poisson bracket* of observables is the operation in local coordinates

$$\begin{aligned}\{F, G\} &= \sum_i \left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial x^i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial x^i}\right) \\ &= \sum_i \mathrm{d}F \wedge \mathrm{d}G\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial x_i}\right)\end{aligned} \tag{9.7}$$

The Poisson bracket plays a fundamental in Hamiltonian mechanics and the relationship between symmetries and conservation laws in physics. This is because (9.7) is a Lie bracket on $C^\infty(T^*M)$. (See [Bry93] for details.)

Suppose that O is some set of observables that is closed under linear combinations, suppose that $\{F, G\} \in O$ for all $F, G \in O$. Then, O is a Lie subalgebra of $C^\infty(T^*M)$ with respect to the Poisson bracket.

Recall that the affine subvariety $\Sigma_N \subset T^*N$ is defined smoothly by observables that take the form of homogeneous functions in the local fiber variables (p_i) of T^*N. For convenience, let us make the additional assumption that the homogeneous functions are algebraic of degree d in (p_i), so that Σ_N is defined smoothly near $\varphi \in \Sigma_N$ by a set of equations in multi-index form

$$0 = F^\varrho(x,p) = \sum_{|I|=d} f^{\varrho,I}(x)p_I, \text{ for } \varrho = \ell+1,\ldots,n. \tag{9.8}$$

Corollary 9.9. *Let O denote the module in $S = C^\infty(N)[p_1,\ldots,p_n]$ generated by (9.8), The eikonal system $\mathcal{E}(\Sigma_N)$ is involutive if and only if $\{O,O\} \subset O$; that is, $\mathcal{E}(\Sigma_N)$ is involutive if and only if the module O is a Lie algebra with respect to the Poisson bracket.*

A proof—which does not depend on the polynomial form (9.8)—can be derived from Corollary 9.5 along with the observation that the Poisson bracket can be defined in a coordinate-free way as the operator such that

$$\{F,G\}(\mathrm{d}\Upsilon)^{\wedge n} = n\mathrm{d}G \wedge \mathrm{d}G \wedge (\mathrm{d}\Upsilon)^{\wedge(n-1)}. \tag{9.10}$$

Finally, note that equations of the form (9.8) appear in analysis as systems of homogeneous first-order PDEs on $u : \mathbb{R}^n \to \mathbb{R}$ of the form

$$0 = F^\varrho(x,p) = \sum_{|I|=d} f^{\varrho,I}(x)\frac{\partial u}{\partial x^I}, \text{ for } \varrho = \ell+1,\ldots,n. \tag{9.11}$$

One famous example is the system of characteristics for the wave equation,

$$0 = -(u_t)^2 + c^2((u_x)^2 + (u_y)^2). \tag{9.12}$$

10. Involutivity of the Characteristic Variety

We would like to apply the entire discussion from Section 9 to the case where Σ_N is the characteristic variety, but first we must establish that Ξ is sensible in T^*N.

Suppose that $\iota : N \to M$ is a connected ordinary integral manifold of an involutive exterior differential system $(M, \mathcal{I})$, and that $M^{(1)}$ is the smooth component of $\mathrm{Var}_n(\mathcal{I})$ containing $\iota^{(1)}(N)$, as in Section 3.

Fix $x \in N$, and suppose $\iota(x) = p \in M$ and $\iota^{(1)}(x) = e \in M^{(1)}$. For $\xi \in \Xi_e \subset V_e^*$, we can consider the pullback $\iota^{(1)*}(\xi) \in \mathbb{P}T_x^*N \otimes \mathbb{C}$. In a basis (η^i) of T_x^*N, we can write a representative as $\xi = \xi_i\omega^i$ for coefficients $\xi_i \in \mathbb{C}$, so that $\iota^{(1)*}(\xi) = \xi_i\eta^i \in \mathbb{P}T^*N \otimes \mathbb{C}$. In this sense, we can pull back the characteristic variety—as a set—to N.

More precisely, recall that Ξ has degree s_ℓ and affine dimension ℓ, but it is a scheme defined by the characteristic sheaf $\mathscr{M}$. For any local section (u_i) of the coframe bundle $\mathcal{F}_{\gamma^*} \to M^{(1)}$, we can write the characteristic sheaf

$\mathscr{M}$ as a homogeneous ideal in the module $C^\infty(M^{(1)})[u_1, \ldots, u_n]$. At each $e = \iota^{(1)}(x) \subset M^{(1)}$, the coframe (u_i) is just a basis of e; therefore, we obtain a basis for T_xN of the form $u_{N,i} = \left(\iota_*^{(1)}\right)^{-1}(u_i)$. To simplify notation, we ignore the subscript N. That is, in some neighborhood of x, the section (u_i) of $\mathcal{F}^*N$ is well-defined. Moreover the stalks of the sheaf $C^\infty(M^{(1)})$ can be pulled back, as $\iota^{(1)*}(f)$ is well-defined for all f defined in a neighborhood of $e = \iota^{(1)}(x)$. Therefore, we can pull back both the coefficients and the coordinates to define the homogeneous ideal $\mathscr{M}_N$ in $C^\infty(N)[u_1, \ldots, u_n]$. Let $\Xi_N \subset T^*N$ be the scheme defined by $\mathscr{M}_N$.

Now, the entire discussion from Section 9 applies where Σ_N is the smooth locus Ξ_N^o of Ξ_N. We know additionally that Ξ_N takes the polynomial form (9.8) as derived from (5.14), so it has degree s_ℓ and dimension $\ell - 1$ at smooth points, as a *complex* projective variety.

Theorem 10.1 (Guillemin–Quillen–Sternberg). *Suppose that N is an ordinary integral manifold of an involutive exterior differential system $\mathcal{I}$ with Cartan character ℓ. The eikonal system of the smooth locus of the (complex) characteristic variety, $\mathscr{E}(\Xi_N^o)$, is involutive. At smooth points in Ξ_N, the characteristic hypersurfaces are parametrized by 1 function of ℓ variables.*

Note that our definition of Ξ_N is the *complex* characteristic variety. This theorem is called the "integrability of characteristics." Cartan demonstrated several examples of this phenomenon in [Car11]. The proof appears in [GQS70], where a major step is the application of Theorem 5.6. Hence, this result appears to rely in an essential way on the facets of the characteristic variety seen in Part II.

The converse is *not* true; it is easy to write down non-involutive exterior differential systems for which $\mathscr{E}(\Xi_N)$ is involutive.

However, in [Gab81], Ofer Gabber proved a more general form of Theorem 10.1 that removes practically all of the technical assumptions and recalls the interpretation of Section 9(b).

Theorem 10.2 (Gabber). *Let R be a filtered ring whose graded ring $gr(R)$ is a Noetherian commutative algebra over $\mathbb{Q}$. Let M be a finitely generated R-module. Then $\{\sqrt{M}, \sqrt{M}\} \subset \sqrt{M}$.*

The previous case occurs when the filtered ring is $\Omega^\bullet(N) \otimes \mathbb{C}$ for an "abstract" integral manifold N, and the finitely-generated R-module M is the characteristic sheaf $\mathscr{M}_N$. By Hilbert's Nullstellensatz, the radical $\sqrt{M}$ plays the role of the module O of functions defining Ξ_N from Section 9(b).

From the general discussion of eikonal systems surrounding Theorem 9.1, the interpretation of these theorems is apparent:

Corollary 10.3. *Suppose that N is an ordinary integral manifold of an involutive exterior differential system $\mathcal{I}$ with Cartan character ℓ. Then N admits a local—possibly complex—coordinate system $(x^1, \ldots x^n)$ such that $\mathrm{d}x^1, \cdots \mathrm{d}x^\ell \in \Xi_N$.*

In [Smi14], the linear span of the characteristic variety, $\langle \Xi_N \rangle$ is studied in comparison to the Cauchy retraction space, $\mathfrak{g}_N^\perp$. Suppose that the affine dimension of $\langle \Xi_N \rangle$ is L and that the affine dimension $\mathfrak{g}_N^\perp$ is ν. These spaces are nested, so $\ell \leq L \leq \nu \leq n$.

COROLLARY 10.4. *Suppose that N is an ordinary integral manifold of an involutive exterior differential system $\mathcal{I}$ with Cartan character ℓ. Then N admits a local—possibly complex—coordinate system $(x^1, \ldots x^n)$ such that $\mathrm{d}x^1, \ldots \mathrm{d}x^\ell \in \Xi_N$, such that $\mathrm{d}x^{\ell+1}, \ldots \mathrm{d}x^L \in \langle \Xi_N \rangle$, and such that $\mathrm{d}x^{L+1}, \ldots, \mathrm{d}x^\nu \in \mathfrak{g}_N^\perp$.*

Corollary 10.4 is a simple result, but its proof relies on building a coframe of N in which the nilpotent parts of the commuting symbol maps B_I^λ are identified clearly; that is, it depends in an essential way on Theorem 10.1 as well as 3.16. The key point is that it reinforces the following dogma:

REMARK 10.5 (General Dogma of the Characteristic Variety). An exterior differential system $(M, \mathcal{I})$ is a geometric object over M, meaning that its key properties are coordinate-invariant. Knowing this geometry is equivalent to knowing the rank-one variety and characteristic scheme, which are prolongation-invariant. Moreover, the geometry of an EDS/PDE imposes a geometry on its solutions, $\iota : N \to M$, and this imposition is dictated by the rank-one variety and characteristic scheme. Therefore, exterior differential systems can be classified up to equivalence as "parametrized families of manifolds N with associated characteristic geometry."

This is not a theorem; it is an attitude.

11. Yang's Hyperbolicity Criterion

One of the great frustrations of the Cartan–Kähler theorem is that it relies on the Cauchy–Kowalevski theorem, so it applies only in the analytic category. One can see its failure in the smooth category in [**Lew57**]. However, this frustration has been escaped in some special cases[18] by exploiting the structure of Ξ.

For example, suppose that $(M, \mathcal{I})$ is involutive over C^∞ and that $\Xi = \emptyset$. Then $\ell = 0$, so the tableau A is the trivial (irrelevant) subspace of $W \otimes V^*$. The prolonged system $\mathcal{I}^{(1)}$ on $M^{(1)}$ is Frobenius, and $M^{(1)}$ is merely a copy of M whose fiber is the unique element is an integrable distribution. That integrable distribution is merely the Cauchy retraction space $\mathfrak{g}$, and it must have been that $\mathcal{I} = \mathfrak{g}^\perp$. The flow-box theorem foliates M by solutions in the smooth category. (Actually, in the Lipschitz category, by standard ODE theory!) If N is a leaf of this foliation, then removing Cauchy retractions on the original exterior differential system $(M, \mathcal{I})$ yields the exterior differential system $(N, 0)$.

Or, for example, suppose that $(M, \mathcal{I})$ is involutive over C^∞ and that $\Xi = V^*$ with $(s_1, s_2, \ldots, s_n) = (r, r, \ldots, r)$. Then, the tableau A is the total space $W \otimes V^*$. Therefore, $M^{(1)}$ is an open domain in $\mathrm{Gr}_n(TM)$, so $\mathcal{I} = 0$,

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[18] If we take the broadest possible interpretation of Remark 10.5 to heart, then any escape from analyticity ought to arise from the structure of Ξ. However, the reader is cautioned again that a dogma is not a theorem.

and there is no condition whatsoever[19] on integral manifolds $\iota : N \to M$; however, the prolongation $\iota^{(1)} : N \to M^{(1)}$ would have to satisfy the contact ideal, forcing some regularity on N. We studied this EDS in Section 2.

A less trivial special case is presented in [Yan87], which is the subject of this section. As it happens, the attempt to understand [Yan87] in the context of [BCG$^+$90, Chapter VIII] was the inspiration for computing the details shown in [Smi15] and the entire approach of these notes.

A tableau A is called *determined* if $s_1 = s_2 = \cdots = s_{n-1} = r$ and $s_n = 0$; that is, $s = (n-1)r$, so $t = r$, and $H^1(A) \cong W$. Cartan's test shows that a determined tableaux is involutive, so we may assume that A is written in endovolutive form as in Theorem 3.16, so the only nontrivial symbol endomorphisms in (1.11) are $\mathrm{B}^\lambda_\lambda = I_{r\times r}$ and B^λ_n for $\lambda = 1, \ldots, n-1$. The quadratic involutivity condition is trivial.

Lemma 11.1. *Suppose A is determined and written in endovolutive bases. Then*

$$\ker \sigma_\varphi = \ker\left(\varphi_\lambda \mathrm{B}^\lambda_n - \varphi_n I\right). \tag{11.2}$$

In particular, $\xi \in \Xi$ if and only if ξ_n is an eigenvalue of $\xi_\lambda \mathrm{B}^\lambda_n$.

Proof. From Part II, we know $w\otimes\xi \in A$ if and only if $\mathrm{B}(\xi)(v)w = \xi(v)w$ for all v. Therefore, we compute in our endovolutive basis

$$\begin{aligned} \xi(v)w &= \mathrm{B}(\xi)(v)w \\ &= \xi_\lambda v^i \mathrm{B}^\lambda_i(w) \\ &= (\xi_\lambda v^\lambda)w + \xi_\lambda v^n \mathrm{B}^\lambda_n w \\ &= (\xi(v) - \xi_n v^n)w + \xi_\lambda v^n \mathrm{B}^\lambda_n w \end{aligned} \tag{11.3}$$

That is, $\xi_n w = \xi_\lambda \mathrm{B}^\lambda_n w$. ☐

Let us identify $H^1(A)$ with W and use our endovolutive basis of W for both. Then $\sigma_\varphi = \left(\varphi_\lambda \mathrm{B}^\lambda_n - \varphi_n I\right)$ for any $\varphi \in V^*$.

Suppose that e' is a *real* hyperplane in e such that $(e')\perp \otimes \mathbb{C} = \varphi \in V^*$ and $\varphi \notin \Xi$. Then $\sigma_\varphi : W \to H^1(A)$ is an isomorphism.

Definition 11.4. Suppose e' is a real hyperplane in e corresponding to the real covector $\varphi = (e')\perp \in \mathbb{P}e^*$. The real hyperplane is called *space-like* if:

(i) $\varphi \otimes \mathbb{C} \notin \Xi_e$, and
(ii) For any $\eta \in \mathbb{P}e^*$, there is a real basis of W in which $(\sigma_\varphi)^{-1}(\sigma_\eta)$ is real and diagonal, and
(iii) that choice of basis is a smooth function of $[\eta] \in e^*/\varphi = (e')^*$.

A determined symbol A is called *determined hyperbolic* if V has a real space-like hyperplane.

Using the expressions above for σ_φ, it is straightforward to verify that this definition holds on a given determined tableau.

[19] The most extreme and amusing exploitations of the flexibility of $\mathrm{Gr}_n(TM)$ come from the homotopy principle [Gro86, EM02].

DEFINITION 11.5. A tableau is called *hyperbolic* if V admits a flag given by a basis $(u^1, \ldots, u^n)$ of V^* such that the sequential initial value problem from $\langle v^i, \ldots, v^n \rangle^\perp$ to $\langle v^{i+1}, \ldots, v^n \rangle^\perp$ has a hyperbolic determined tableau.

THEOREM 11.6 (Yang). *Theorem 3.15 applies in the smooth category, if A is hyperbolic.*

The proof proceeds by replacing the Cauchy–Kowalevski initial-value problem with the Cauchy initial-value problem for determined first-order quasilinear hyperbolic PDEs. See [Yan87] and Appendix A of [Kam89] for more details.

Clearly the definition of *hyperbolic* depends on the geometry of Ξ and the symbol maps B^λ_i; however, to my knowledge no one has succeeded in writing down the explicit condition on B^λ_i or $\mathscr{C}$ or Ξ for general hyperbolicity. Hence, Yang's condition is not yet available to computer algebra systems. If that can be accomplished, it means we can identify a subvariety of the moduli of involutive tableaux—as in Section 3(d)—that admit solutions in the smooth category.

One well-understood special case is when $\ell = 1$, so Ξ_e contains s_1 real points (with multiplicity). If the number of distinct points is sufficiently large (greater than n), then this is the situation for hyperbolic systems of conservation laws, as in [Tsa91]. The eikonal system is rigid, so each solution is foliated by s_1 characteristic hypersurfaces. Multiplicity corresponds to nilpotent pieces of the generalized eigenspaces of the symbol endomorphisms B^1_i.

12. Open Problems and Future Directions

Our perspective here has been simple-minded, to gain intuition of Ξ and $\mathscr{C}(\Xi)$ as rapidly as possible. The articles [Smi14] and [Smi14] are founded on this perspective, but reveal additional detail of the structures discussed here. For more modern and sophisticated treatment, please see [Mal03], [KL07], and [Car09]. Additionally, Chapters V–VIII of [BCG$^+$90] contain significantly more results than we have summarized here.

To conclude, here are some interesting questions which—to my present knowledge—are open subjects of ongoing research.

(i) Moduli of Involutive Tableaux. Section 3(d) demonstrates a first step toward understanding the variety of involutive tableaux. Within this variety, can we identify the sub-variety of hyperbolic tableaux?, of elliptic tableaux? of integrable systems? Where do the Lewy example fall in this variety? If there is any organizing geometry behind the impenetrable jungle of involutive PDEs, this is where we should look.

(ii) Global Integrability of the Characteristic Variety. If A is involutive, then the system $\mathscr{C}(\Xi_N)$ is involutive on an ordinary integral manifold, N. However, is there a clear sense in which Ξ is involutive over $M^{(1)}$ itself? That is, consider the EDS on $M^{(1)}$ generated by adding a smooth section ξ to $\mathcal{I}^{(1)}$. Under what circumstances is this involutive? Can Gabber's theorem be adopted to this case?

(iii) Can Gabber's theorem provide integrability results for PDEs with low regularity?

(iv) The Prolongation Theorem. Does prolongation always work, if we remove the regularity assumptions on $M^{(1)}$ and consider the many components of $\mathrm{Var}_n(\mathrm{Var}_n(\cdots(\mathcal{I})\cdots))$?

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