WHAT IS... GUILLEMIN NORMAL FORM?

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To many young students, *linear maps* seem absurdly complicated; the only way to access their properties is through intricate and unintuitive procedures that appear to solve specific problems, but never get at the question "what *are* linear equations? How can I organize them?" If it's hard to remember this feeling, it's because you already know Jordan normal form, a framework to understand all the essential properties of a linear map without worrying about its original presentation in a basis.

To many working mathematicians *differential equations* seem absurdly complicated; the only way to access their properties is through intricate and unintuitive procedures that appear to solve specific problems, but never get at the question "what *are* differential equations? How can I organize them?"

This is an introduction to Guillemin normal form, one of the most beautiful and overlooked structures in 20th century mathematics, which may offer a framework to understand many of the essential properties of differential equations. It reveals a profound connection between algebra, geometry, and analysis.

1. Generalizing Jordan

Let W be an r-dimensional complex vector space, so $\mathrm{End}(W)$ is the space of linear maps or $r \times r$ matrices on W. For any matrix $X \in \mathrm{End}(W)$, we can apply Jordan form, decomposing W according to the Jordan blocks of X. If $0 \neq c \in \mathbb{C}$, then the matrix cX yields the same Jordan decomposition of W as X. Thus, Jordan form can be interpreted as a classification of lines in $\mathrm{End}(W)$ or points in $\mathbb{P}\,\mathrm{End}(W)$. We can generalize both lines and points.

Lines in $\mathrm{End}(W)$ can be generalized to linear subspaces. Say $X, Y \in \mathrm{End}(W)$. If X and Y are arbitrary, then they may have completely different Jordan forms. But, recall that X and Y admit compatible Jordan blocks if and only if $XY = YX$. Thus, to generalize Jordan form, we want to consider s-dimensional linear subspaces $\mathfrak{g} \subset \mathrm{End}(W)$ of commuting maps. The moduli space of these *commutative matrix algebras* is already of interest to algebraists, but to study PDEs, we need varying families of them.

Points in $\mathbb{P}\,\mathrm{End}(W)$ can be generalized to varieties. Let Ξ be an algebraic variety defined by a ideal $\mathcal{M}$ on some space V^*, and where each $\xi \in \Xi$ (being careful with components and multiplicity) corresponds to a subspace $\mathfrak{g}(\xi) \subset \mathrm{End}(W)$. Moreover, we insist that each $\mathfrak{g}(\xi)$ is commutative, allowing simultaneous Jordan decomposition for all $X \in \mathfrak{g}(\xi)$.

Therefore, we have a purely algebraic object, constructed from the familiar ideas of endomorphism, eigenvector, and variety. If you're comfortable with these topics, one more generalization is possible: Given a manifold or variety N, we can suppose that $\mathcal{M}$ is a sheaf over N, so that $\mathcal{M}$ and Ξ vary with points in N. For this article, let's call these spaces and maps

$$\begin{cases} 0 \to \mathfrak{g}(\xi) \to \mathrm{End}(W) \to \mathfrak{g}(\xi)^{\perp} \to 0 \\ \xi \in \Xi = \mathrm{Var}(\mathcal{M}) \subset V^*, \text{ over } N \end{cases}$$

a "commutative characteristic system."

2. Differential Equations

When studying differential equations, the primary question is "Can we solve the initial-value problem? In how many ways?" That is, we want to parametrize the moduli of solutions. This question is solved when one can establish *involutivity*, a formal property whose lengthy definition (see [3]) says that one can integrate the variables one-by-one without needing higher derivatives. The Cartan–Kähler theorem: "If a system of PDEs is involutive, then it admits a family of series solutions depending smoothly on arbitrary choice of s_ℓ functions of ℓ variables." The numbers ℓ and s_ℓ are computed while checking for involutivity and provide a Hilbert polynomial.

The Cartan–Kähler theorem has a partial converse, the Cartan–Kuranishi prolongation theorem: "If a system of PDEs admits a smooth family of solutions, then some finite prolongation of the system is involutive." Prolongation is the familiar process in which one includes derivatives as new variables in a system of PDEs. For example, $y = y''$ becomes $\{p = y', y = p'\}$. This process is canonical and coordinate-free when interpreted on the Grassmann using differential forms.

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Therefore, when constructing solutions of PDEs, we usually restrict our attention to involutive PDEs. (Actually, these theorems apply only in the analytic category and a few other specialized cases, as they require the Cauchy–Kowalevski theorem to produce a sequence of power series. Since the systems described here are algebraic, this limitation is moot.)

Suppose you have a system of PDEs in n independent and r dependent variables. Complexifying and writing $V = \mathbb{C}^n$ and $W = \mathbb{C}^r$, the *tableau* of the PDE is the space $A \subset W \otimes V^*$ of all partial derivative Jacobian matrices that appear in solutions. The annihilator of A is given by a linear map called the *symbol* of the PDE, σ:

$$0 \to A \to W \otimes V^* \xrightarrow{\sigma} A^\perp \to 0.$$

Solutions to the initial-value problem are controlled by the *characteristic variety*, $\Xi \subset V^*$, which is defined by the property that the infinitesimal initial-value problem of the PDE is under-determined on the hyperplane $\xi^\perp \subset V$. The geometry of Ξ is the source of the phenomena of elliptic, hyperbolic, and parabolic systems in 2 dimensions, but it is much wilder for larger n.

Guillemin studied these tableaux and characteristics using modern algebra [1]. For each $\xi \in \Xi$, one can construct a map $B_\xi : V \to \mathrm{End}(W)$ using the components of the symbol map σ. Omitting details, it can be written something like

$$B_\xi(v) : w \mapsto \sigma(w \otimes \xi) \cdot v.$$

Let $\mathfrak{g}(\xi) = B_\xi(V) \subset \mathrm{End}(W)$. Among Guillemin's results is "Suppose that the system of PDEs in involutive. For each $\xi \in \Xi$, the algebra $\mathfrak{g}(\xi)$ is commutative." That is, the $B_\xi(v)$ admit a simultaneous Jordan form that varies with $\xi \in \Xi$, and one can choose clever bases for V, W, and A so that the involutive PDE looks as nice as possible. This is Guillemin normal form, a sheafy, varying generalization of Jordan form. In our language, an involutive PDE yields a commutative characteristic system.[1]

3. Meaning of Moduli

Is this algebraic perspective useful? Jordan form reveals structures of linear maps; likewise, Guillemin form reveals structures of differential equations. For example: An involutive PDE has *constant coefficients* if N is a point. An involutive PDE is an *ODE* if Ξ is trivial. An involutive PDE is *elliptic* if Ξ is strictly complex. But, we should hope for more. Here are three fundamental questions in PDE analysis whose answers should be algebraic:

3.1. **Involutivity.** A completely open question is to describe the moduli space of involutive PDEs; that is, among all PDEs, describe those that are prolonged enough to admit families of power-series solutions. Even in the case of PDEs with constant coefficients and a fixed Ξ, the problem is very difficult beyond 2 or 3 variables. As the moduli of commutative algebras becomes better understood, this generalization becomes more practical.

3.2. **Hyperbolicity.** The most severe limitation of Cartan–Kähler theory is its reliance on analyticity, but in the special case of hyperbolic systems, one can solve PDEs over $\mathbb{R}$ in the *smooth* category. However, the technical definition of hyperbolic systems is very difficult to verify in general, so the hyperbolic theory has been applied only in a handful of examples.

3.3. **Integrability.** The word "integrability" means many things to many people, usually focused on a small set of favorite wave-like PDEs and techniques. A more general notion, applicable in all dimensions, is needed greatly. Integrability phenomena and special coordinate systems for PDEs frequently involve the existence of families of solutions that intersect maximally with the characteristic variety and related Veronese and secant varieties, so it should be detectable via this sort of algebraic perspective.

One might hope that these three questions are related tightly: A decomposition of the variety of involutive tableaux over $\mathbb{R}$ should yield hyperbolic systems as a subvariety, and integrable systems might appear as a further subvariety of those hyperbolic systems. The investigation of these three ideals promises fascinating interactions between algebraic geometry, differential geometry, and analysis.

Like Jordan form prompted representation theory for finite-dimensional Lie groups, Guillemin form allowed progress in the theory of infinite-dimensional Lie pseudogroups [2]. More broadly, the application of Jordan form yielded major parts of functional analysis, quantum mechanics, statistics, and many other disciplines. The mind boggles at the possible applications of a comprehensive algebraic classification of involutive PDEs.

[1] Interestingly, the converse is not immediately clear. To my knowledge, no one has studied *which* abstract varieties can be characteristic varieties, or whether there is some additional condition imposed on those varieties arising from PDEs.

References

[1] V Guillemin, *Some algebraic results concerning the characteristics of overdetermined partial differential equations*, Amer J of Math (1968), 270–284.

[2] V Guillemin, D Quillen, and S Sternberg, *The integrability of characteristics*, Comm on Pure and Appl Math **23** (1970), no. 1, 39–77.
[3] T Ivey and JM Landsberg, *Cartan for Beginners*, AMS GSM. 61, 2003.

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